Discrete Choices under Social Influence: Generic Properties

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Abstract

We consider a model of socially interacting individuals that make a binary choice in a context of positive additive endogenous externalities. It encompasses as particular cases several models from the sociology and economics literature. We extend previous results to the case of a general distribution of the idiosyncratic preferences, the Individual Willingnesses to Pay (IWP).

Positive additive externalities yield a family of inverse demand curves that include the classical ones but also new ones with non constant convexity. When j, the ratio of the social influence strength to the standard deviation of the IWP distribution, is small enough, the inverse demand is a classical monotonic (decreasing) function of the adoption rate. Even if the IWP distribution is mono-modal, there is a critical value of j above which the inverse demand is non monotonic, decreasing for small and high adoption rates, but increasing within some intermediate range. Depending on the price there are thus either one or two equilibria.

Beyond this first result, we exhibit the *generic* properties of the boundaries limiting the regions where the system presents different types of equilibria (unique or multiple). These properties are shown to depend *only* on qualitative features of the IWP distribution: modality (number of maxima), smoothness and type of support (compact or infinite). The main results are summarized as *phase diagrams* in the space of the model parameters, on which the regions of multiple equibria are shown.

We investigate the generic consequences of these properties on a monopolistic market. For a large enough value of j, the optimal strategy for the monopolist exhibits a drastic jump from one with a high price and a low number of customers, to one of low price and a large number of customers. This discontinuity arises even for values of j where the demand is uniquely defined.

Contents

1 Introduction

1.1 Modeling social influences

There are many circumstances in social and economic contexts where, faced with different alternatives, the best choice for an individual depends on the choices of other individuals in the population. The decision of leaving a neighborhood [77], to attend a seminar [77] or a crowded bar [2, 3], to participate to collective actions such as strikes and riots [44], are particular examples taken from social sciences. It has been suggested that social interactions may explain the school dropout [23], the persistence in the educational level within some neighborhoods [29] and the related consequences in the stratification of investment in human capital and economic segregation [11], the large dispersion in urban crime through cities with similar characteristics [41], the emergence of social norms [71], the labor market behavior and related unemployment patterns [87, 85], the housing demand [95], the existence of poverty traps [33], the smoking behavior [54, 53, 82], etc.

Similarly, there is a growing economic literature that recognizes the influence on consumers of the social world they live in. In market situations like the subscription to a telephone network [5, 74, 89, 24] or the choice of a computer operating system [50], the willingness to pay generally depends not only on the individual preferences but also on the choice made by others [80, 75]. If the externality is positive the utility for the most popular choice increases even for individuals who otherwise would never make this choice. In other words, the conformity effect may dominate the heterogeneity of preferences, as pointed out by Bernheim [12]. General aspects of these issues have been discussed in the literature [10, 60]: particular insightful papers are Becker's note [9] about restaurants pricing, and the qualitative analysis of the consequences of interpersonal influences ("bandwagon effects") on the consumers demand and on the supply prices, by Granoveter and Soong [45]

In the present paper we consider the general properties of a model of socially interacting individuals that make a binary choice in a context of positive endogenous externalities. The model encompasses, as particular cases, most of the above mentioned models presented in the sociology and economics literature. We explore the consequences of the externalities on the economy, taking as an example the simplest market, i.e. that of a monopolist pricing a single good.

In social sciences, the question of discrete (typically binary) choices with heterogeneous agents and positive externalities has been first addressed in the 70's by Schelling [76, 77], who borrowed from Physics the concept of *critical mass*: in a repeated-decisions setting, depending on whether this critical mass is or not reached, the system may end up at very different equilibria. Granovetter further develops Schelling's model, applying it to particular problems such as joining or not a riot [44], voting, etc [45]. The same topic is reconsidered within a statistical physics point of view in the early 80's by Galam *et al* [38]. The notion of critical mass is then associated to the Physics concept of *phase transition* at a critical point, in the neighborhood of which the system may be extremely susceptible: by tipping effects, small microscopic changes can lead to drastic changes at the macroscopic level. Similar tools have been applied in 1980 by Kindermann and Snell [51] to the study of social networks. These authors introduced into the sociology and economics literature the equivalence between statistical physics approaches –that use the Boltzmann-Gibbs distribution– and Markov Random Fields. Another physically-inspired approach for modeling social phenomena such as opinion diffusion has been developed by Weidlich and Haag [92, 91] in 1983, through a master equation and the Fokker-Planck approximation. Later, these physically inspired models of opinion contagion have been exploited in economics by Orléan [69, 70] for the analysis of mimetic behaviors in the context of financial markets. There is now a large and growing literature on opinion and innovation diffusion (see e.g. [88, 47, 25, 90, 83]) closely related to the general discrete choice model considered in this paper. Since the beginning of the 90's the general framework of social interactions in non-market contexts is reconsidered in a Beckerian way [8, 10], in particular by Glaeser *et al* [40, 41, 39].

The first application of statistical mechanics approaches in economics may be traced back to the pioneering work of Fölmer [35]. Introducing an economic interpretation of the Ising model of ferromagnetism at finite temperature, he shows that strong externalities may hinder the stabilization of an economy. These models introduce Markov random fields (equivalently Boltzmann-Gibbs distributions) to model uncertainty in the decision making process, allowing for the definition of a general equilibrium concept. According to Fölmer, Hildebrand's [46] justification of the representative agent approach breaks down when agents' decisions are correlated due to their social interactions (for a discussion, see also [52]).

A renewal of interest for models of binary decisions with externalities arose in economics in the 90's. On one side, Durlauf and collaborators [26, 27, 28, 29] and Kirman and Weisbuch [93] among others, consider agents that choose an action according to a Boltzmann-Gibbs distribution, that is a *logit* choice function, reflecting some random aspects in the agent's utility. In this context Brock [18] and Blume [13, 14] explicit the links between Game Theory and Statistical Mechanics, while Kirman and coworkers [67] show that the logit choice function may be seen as resulting from an exploration-exploitation compromise. These and other recent papers [4, 30, 31, 19, 20, 39, 94, 34, 43, 65] analyze with statistical physics tools the consequences of positive social (market and non-market) interactions in the aggregate behavior of large populations (for a short introduction to statistical physics approaches see [42] and for their application to economics see [72]; see also [7] for a survey).

The mentioned authors restrict the analysis of the model to the case where all the individuals have the same idiosyncratic preference. Heterogeneity in the population is introduced through the probabilistic decision-making process, like in Fölmer's work. Then, the actual equilibrium reached by the system depends on the fixed points of the decision dynamics, generally a myopic best reply. An interesting characteristic of these models is that they present multiple equilibria for some range of the parameters. Becker [9] pointed out important consequences of these multiple equilibria, induced by externalities, on the economy: he suggests that they could be the reason of seemingly suboptimal pricing in situations of persistent excess demand.

In this paper we consider intrinsically heterogeneous agents with fixed utilities, like in McFadden's approach to Quantal Choice models [63]. We mainly (but not exclusively) decline the model within a market context: the binary choice being to buy or not a given good at a price posted by a monopolist who determines the price in order to maximize his profit. The (fixed) individual willingnesses to pay (IWP) are randomly distributed among the population according to a given probability density function (pdf). This general setting allows us to generalize Becker's qualitative analysis of the optimal pricing problem. Putting the price to zero allows us to recover the social sciences models.

We determine the possible equilibria of the system without assuming any precise decision making dynamics. We show that the model properties depend on the strength of the externality and on qualitative properties of the willingness-to-pay (IWP) pdf, like its modality class (the number of maxima), its smoothness properties and the kind of support. We display the main results on a plane whose axes are the parameters of the model, namely, the average IWP and the strength of the social component, both measured in units of the standard deviation of the IWP distribution. We plot the boundaries of the regions where different types of solutions exist, a representation called *phase diagram* of the system. The particularly important case of a uni-modal pdf (with a single maximum) is thoroughly studied, but we also discuss the consequences of multi-modality.

Particular cases of the model were presented elsewhere [66], and a specific case has been treated in details in [43]. This paper extends those result to the case of a general IWP distribution.

1.2 More on related models

In this section we briefly discuss the relationship between the model to be considered here and other models studied in the literature. Let us first consider models of discrete choices in the absence of externalities. According to the typology proposed by Anderson *et al.* [1], within the general framework of *Random Utility Models* (RUM)[62, 59] with *additively* stochastic utilities, there are two distinct approaches to individual choices: a "psychological" one and an "economic" one. In the psychological perspective (Thurstone [86], Luce [58]) the randomness is a time-dependent i.i.d. random variable: the random components of the idiosyncratic preferences are assumed to be independently drawn afresh by each individual from a given pdf, each time the choice has to be made. They are interpreted as individual temporary changes, or mistakes in the estimated utilities. In the simplest case –actually, the only one treated in the social and economic literature– the agents IWPs have identical deterministic parts and only differ by this random time-varying term which is systematically assumed to be drawn from a logistic pdf. In practice many approaches like in [19] consider the choice rule as deriving from a random utility model [58]. As shown by McFadden [62], in this context the logistic form is obtained if the random terms in the underlying Thurstone's discriminant process are i.i.d. Weibull random variables, i.e. have a double exponential (extreme value, type I) distribution (see also [1]).

In the presence of strategic complementarities ([21, 22]), the resulting model is well known in statistical physics: it corresponds to the *standard Ising model* i.e. with ferromagnetic interactions and *annealed* disorder, that is, at finite temperature T. The latter is the inverse of the standard logit parameter $\beta \equiv 1/T$ and is thus proportional to the standard deviation of the IWP distribution. The ferromagnetic interaction constant J corresponds to the strength of the social externality. Introduced by Fölmer [35] in the economics context, this standard Ising model has recently been reconsidered in the social and economic literature mainly by Durlauf and coworkers [30, 31, 19, 20] and by Kirman and Weisbuch [93, 67]. The corresponding equilibria are reminiscent of the Quantal Response Equilibria [64] used in the context of experimental economics and behavorial game theory. These are equilibria "on the average", in the statistical sense (as in Physics): they do not correspond to the strict maximization of the utilities (that are random variables) but to their estimated or expected values. In the generally considered infinite population limit (where the variance of the expected values vanishes) the expected utilities are systematically smaller that the maximal ones.

The standard Ising model is quite well understood [84]. In the case where the agents are situated on the vertices of a 2-dimensional square lattice, each having four neighbors, there is an analytical description of the stationary states of the model, due to Onsager [68]. However, no analytic results exist for arbitrary neighborhoods except for the specific case of a global neighborhood, known as the *mean field* approximation in Physics. Considering global neighborhoods, Brock and Durlauf [19] analyze the properties of the expected demand in the case of subjective (rational) expectations under the assumption of a logistic distribution of such expectations (assuming thus double exponential random utilities). They find, in agreement with standard results in statistical mechanics [84], that there exist either one, two or three solutions for the demand function, depending on the relative magnitudes of the idiosyncratic uniform social term, the variance of the stochastic term and the strength of the social effects.

In the following we adopt instead McFadden's [62] economic approach (see also Manski [59, 1]: we assume that each agent has a willingness to pay *invariable* in time, that is different from one agent to the other. In statistical physics this heterogeneity is called *quenched* disorder. The particular model we study is analogous to the ferromagnetic *Random Field Ising Model* (RFIM) at zero temperature (corresponding to the fact that the agents make deterministic choices). Thus, our modeling approach assumes the so called "risky" situation: an external observer (e.g. a seller) does not have access to the individual preferences, but may know their probability distribution. According to McFadden [62, 63], "Thurstone's construction is appealing to an economist because the assumption that a single subject will draw independent utility functions in repeated choice settings and then proceed to maximize them is formally equivalent to a model in which the experimenter draws individuals randomly from a population with differing, but fixed, utility functions, and offers each a single choice; the latter model is consistent with the classical postulates of economic rationality" ([63], p 365). However, in the presence of social interactions this statement is in general incorrect: first, in a repeated choice setting, individual utilities evolve in time according to the others' decisions, hence the time average on a single agent and the population average at a given time do not necessarily coincide; second, it is known from the statistical physics literature that the equilibrium properties of systems with quenched disorder differ from those with annealed disorder. The properties of quenched disordered systems have been and still are the subject of numerous studies in statistical and mathematical physics. Since the first studies of the RFIM, which date back to Aharony and Galam [37, 36], a number of important results have been published in the physics literature (see e. g. [79]). Several variants of the RFIM have already been used in the context of socio-economic modeling, both by physicists and economists [38, 70, 17, 94]. In contrast to the standard Ising model at finite temperature, the properties of the RFIM with externalities, at zero and at finite temperatures, are far from being fully understood. In particular, it is important to emphasize that the equilibria of systems with annealed and quenched disorder, reached through a dynamics that corresponds to an iterated game where agents make myopic choices at each time-step, may be of very different nature.

1.3 Main specific results

In the present paper we determine the *equilibrium properties* in the case of a *global neighborhood* with time invariant (quenched) random utilities, in the limit of an infinite number of agents. Since our paper focuses on equilibrium (static) properties, the social influence depends on the *actual* choices of the neighbors, in contrast with [19], where the social influence in the surplus function depends on the agent's *expected* demand.

The quenched-utilities model (RFIM at zero temperature) and the standard mean field Ising model at finite temperature (annealed disorder) have the same aggregate behavior (i.e. demand function for the market case)and equilibria under the following conditions - but essentially *only* under such conditions:

1. the choice function with annealed utilities is identical to the cumulative distribution of the quenched IWPs;

2. in the annealed case, equilibrium is reached through repeated best reply choices, where the expected demand is myopically estimated;

3. the population size is infinite, guaranteeing that the variance of the demand vanishes in both models.

However, the economic interpretation of these equilibria are very different: in the case of quenched utilities these are standard Nash equilibria, while in the case of annealed utilities these are similar (although not identical) to Quantal Response equilibria.

Most previous studies using annealed or quenched utilities consider specific probability distributions, mostly a logit or a Gaussian [86]. Some papers have determined conditions on the choice function for having multi-equilibria [67, 39]. From the Physics literature we expect that specific properties near a *critical point* (a bifurcation point, see section 3.2.1 below) are independent of the details of the model: this is used in [79] for describing the hysteresis effects in a family of (physical) systems at such a critical point, and exploited in [65] for the analysis of empirical socio-economic data in cases where the actual pdfs are not known. However, the full description of the *phase diagram* for an arbitrary pdf has not been done yet. Here we present this detailed analysis for a typical probability distribution of the IWP. We show how uniqueness or multiplicity of equilibria, related to convexity properties of the inverse demand functions, result from modality and smoothness of the pdf, as well as from the strength of the externality.

We show that for a small enough social influence (the case of moderate social influence in [39]), the demand has a classical shape, that is with a continuous decreasing adoption rate for increasing prices. Beyond a critical value of the ratio between the strength of the social influence and the standard deviation of the IWP distribution the inverse demand function exhibits a non classical behavior –and this even if the distribution of preferences is mono-modal–: the inverse demand presents a non monotonic behavior. As a result, depending on the price, there are either one or two stable equilibria for the demand: the positive (additive) externalities in a market context may give raise to a family of non-monotonic demand curves generalizing thus the classical ones.

Beyond this first main result, we exhibit the generic properties of the boundaries limiting the regions where the system presents different types of equilibria (unique or multiple). We call these properties *generic* since we show that they depend *only* on qualitative features of the IWP distribution: modality (number of maxima), smoothness (continuity and derivability properties) and type of support (compact or infinite). The main results are summarized as *phase diagrams* in the space of the model parameters, exhibiting in particular the regions of multiple equilibria. Here the relevant parameters are (i) the social influence strength and (ii) the difference between the population average of the IWP and the posted price, both normalized by the standard deviation of the IWP distribution (the importance of the heterogeneity).

We extend this analysis to the case of a monopolistic market, considering the supply-demand equilibrium. In the study of the monopolist's optimization problem, an interesting mathematical structure emerges. An effective supply function may be defined such that, at least formally, the analysis of the supply-demand equilibrium relies on the very same mathematical structure as the one involved in the analysis of the demand function. Consequently, the monopolist's optimization presents a phenomenon analogous to the one existing in the demand: there is a critical value of the social influence strength beyond which there are multiple relative maxima of the profit for a finite range of the model parameters. Within this range, there are particular values of these parameters where the optimal strategy exhibits a drastic jump from one with a high price and a small fraction of customers, to another with a low price and a large fraction of customers. Through a qualitative analysis on the demand, the existence of these two possible strategies has been put forward by Becker [9] to explain seemingly irrational pricing. Interestingly, we show that the critical social influence strength beyond which the monopolist has to change his pricing strategy is *not* the same as the one beyond which the customers system presents multiple solutions, but it is strictly smaller.

Like for the demand, these properties are *generic*: they are the same for IWP distributions sharing the same characteristics concerning modality, smoothness and type of support. The monopolist's phase diagram summarizes these results in the space of parameters which are, here, the social influence strength and the average over the population of the IWP, both normalized by the standard deviation of the IWP distribution.

1.4 Organization of the paper

The paper is organized as follows. In the next section 2 we present the model: in section 2.1 we specify the agents (customers) model, in section 2.2 we introduce a normalized form of the basic equations which is convenient for analyzing the demand, and in section 2.3 we show on two simple extreme cases what to expect from these equations. In section 2.4 we detail the families of probability distributions covered by this paper.

In section 3 we analyze the aggregate demand (the collective behavior) for a generic smooth pdf. In section 3.1 we introduce and study the direct and inverse demand functions, and derive the demand phase diagram in section 3.2: we obtain the domain of multiple solutions section 3.2.1, which allows to plot the phase diagram, section 3.2.2, and we discuss some details : the vicinity of the bifurcation point, section 3.2.3, and the question of Pareto optimality, section 3.2.4. Finally a summary of the demand properties is given in section 3.3.

The Supply side is studied in section 4. First the monopolist model is presented in section 4.1; the profit maximization problem for the monoplist is formalized section 4.2, with some technical aspects left to Appendix A. Then the Supply phase diagram is derived and analyzed in 4.3: the multiple solution region is obtained in section 4.3.1, the line of null price in section 4.3.2, and all the results are presented on the phase diagram in section 4.3.3. Finally, the main results for the supply side are summarized in section 4.4.

In Appendix B, we study the demand and supply phase diagrams of IWP distributions with compact support (section B.1) and with fat tails (section B.2); the demand phase diagram for a pdf with an arbitrary number of maxima is studied in B.3 – in details for a smooth multimodal pdf in section B.3.1, and on a simple example of singular bimodal distribution in section B.3.2.

Finally the main results and perspectives are given in section 5.

2 Model of discrete choices with heterogeneous agents and positive externalities

2.1 Agents model

We consider a population of N agents ($i = 1, 2, \ldots, N$). Each individual i has to make a binary choice. Depending on the context, this binary decision may represent the fact of buying or not a good, adopting or not a given standard, adopting or not some social behavior such as joining a riot [44], or a journal club [76], [77], etc. Formally each agent i must choose a strategy ω_i in the strategic set $\Omega = \{0, 1\}$[1]. Hereafter, without loss of generality, we will refer to the simplest market situation where the agents are customers who must choose whether to buy or not a single good at a price P. In this section and the next one (sections 2 and 3), our main concern is with the agents' behaviors, and P is considered as an exogenous parameter —e. g. it is posted by a monopolist selling the good—. Non-market models like those recently considered by, i.e., Glaeser et al. [39] are obtained by setting $P = 0$ or by considering P as an exogenous social cost, common to all the agents. We are interested in the collective outcome of the agents decisions. Next, in section 4, focusing on the market context we will analyze the consequences of the customers collective behavior on the monopolist's program for fixing the optimal price.

The population is heterogeneous. Each individual i has an *idiosyncratic* preference or *willingness to pay/adopt* (hereafter IWP) H_i, meaning that in the absence of social influences, an agent i adopts the state $\omega_i = 1$ if H_i is larger than the price P. Following Mc-Fadden [62] and Manski [59], we work within the framework of *Random Utility Models* (RUM): we assume that H_i is a random variable independently and identically distributed (i.i.d.) in the population. Denoting by H the mean and by σ the variance of the IWP distribution, hereafter we assume that the random variable $(H_i - H)/\sigma$ is distributed according to:

$$\mathcal{P}(x < \frac{H_i - H}{\sigma} < x + dx) = f(x)dx, \tag{1}$$

so that f is a pdf with zero mean and unit variance. In section 2.4 below, we present in details the class of pdfs considered in this paper.

If all the individuals had the same IWP, the outcome would be very simple: either the price is below this common value, and everybody buys, or it is higher than it and nobody buys. All the individuals would behave in the same way, and in the market aggregate analysis, they may be replaced by a fictitious *representative agent* [52]. In the case of a heterogeneous population considered here, only the agents with $H_i \geq P$ would buy at price P.

The situation is more complex when the decision of each agent depends *also* on the decisions of others [59] and references therein. We assume that each agent is the more willing to pay the larger the number of buyers in the population. We consider a linear separable surplus, that is *if* agent i buys at the posted price P, his surplus is

$$S_i = H_i + J\eta - P, \tag{2}$$

where η is the fraction of buyers in the population

$$\eta \equiv \frac{1}{N}\sum_{i=1}^{N} \omega_i. \tag{3}$$

The externality $J\eta$ corresponds to strategic complementarities, that is, the strength of the social influence is positive: $J > 0$[2].

[1] some authors use the notation $s_i = 1$ and $s_i = -1$; both encodings are equivalent: it suffices to replace $\omega_i = (s_i + 1)/2$ in our model and identify the coefficients of corresponding expressions.

[2] More generally, the social term may be proportional to the fraction of buyers in an individual-depending

In order to maximize his surplus, agent i should buy/adopt ($\omega_i = 1$) if $S_i > 0$, but not ($\omega_i = 0$) when $S_i < 0$. Thus, his actual surplus is

$$W_i = S_i \, \omega_i. \tag{4}$$

Since the IWP are i.i.d., when N is very large (more precisely, in the limit $N \to \infty$), by the law of large numbers, the fraction of buyers (3), which is the average of ω_i, converges to the expected value of ω_i over the IWP distribution. Thus, η is given by the fixed point equation:

$$\eta = \mathcal{P}(H_i - P + J\eta > 0). \tag{5}$$

The marginal customer m, indifferent between adopting or not, is defined by the condition of zero surplus, $S_m = 0$:

$$H_m - P + J\eta = 0 \tag{6}$$

so that (5) may be written as

$$\eta = \mathcal{P}(H_i > H_m). \tag{7}$$

For what follows it will be more useful to write (5) as

$$\eta = \mathcal{P}(H_i - H > S) \tag{8}$$

where

$$S = S(J, H, P;\ \eta) \equiv H - P + J\eta \tag{9}$$

is the population average of the (*ex ante*) surplus S_i. Our notation $J, H, P; \eta$ indicates that J, H and P are considered as parameters, whereas η is the variable. Among the parameters, P is the exogenous price, while J and H are properties of the customer population.

2.2 Aggregate behavior: normalized equations

Clearly, the fraction of buyers η depends on the strength of the social influence J, the price P and the average willingness to pay in the population H, and on the distribution of the deviation of the IWP H_i from its population average H. The agents choices depend only on the surplus sign, and they are invariant under changes of the surplus scale. Since the surplus is linear, we can formally multiply every term of the surplus by a same strictly positive number without changing the agents choices. An adequate scale is given by the typical scale of the IWP distribution: it is convenient to measure each quantity (J, H, P) in units of the width σ of the IWP pdf. Hence instead of four parameters, one is left with three independent parameters.

Hereafter we will thus work with the following normalized variables

$$j \equiv \frac{J}{\sigma}, \quad h \equiv \frac{H}{\sigma}, \quad p \equiv \frac{P}{\sigma} \tag{10}$$

In addition, as it is obvious from equations (8) and (9), η depends on the price P and the average willingness to pay H only through their difference $H - P$. We introduce the normalized difference:

$$\delta \equiv \frac{H - P}{\sigma} = h - p, \tag{11}$$

which is the average ex-ante surplus in the absence of externality. For short hereafter we call δ the *bare surplus*. In non-market models ($p = 0$) it is the average willingness to adopt.

Remark: In (almost) all the following we will work with the above reduced variables (10), (11), referring to them as the (normalized) strength of social influence, average willingness to

subset of the population, called "neighbors" of agent i. In this paper, we consider a global neighborhood, where every agent has social connections with every other agent, mainly because it can be studied analytically.

pay, price, and bare surplus. However one should keep in mind, especially when interpreting the results, that they represent the *ratios* of these (non normalized) parameters to the width of the IWP distribution. Clearly other normalization would be possible. An alternative of particular interest is the normalization obtained by measuring every quantity in units of the social strength J: the relevant parameters are then

$$\tilde{\sigma} \equiv \frac{\sigma}{J}, \quad \tilde{h} \equiv \frac{H}{J}, \quad \tilde{p} \equiv \frac{P}{J}, \quad \tilde{\delta} \equiv \frac{H-P}{J} = \tilde{h} - \tilde{p} \tag{12}$$

(equivalently one can do as if $J = 1$). Note that this choice of normalization is no more than an equivalent representation of the parameters space; indeed one has $\tilde{\sigma} = 1/j, \quad \tilde{h} = h/j, \quad \tilde{p} = p/j$. However it will also be interesting to analyze the results in term of this set of parameters, which amount to consider the model properties in function of the strength of the heterogeneity (relative to a given strength of the social influence) - an homogeneous population corresponding to the limiting case $\tilde{\sigma} = 0$, a highly heterogeneous one to a high $\tilde{\sigma}$ value.

With the normalized variables (10), (11), equation (8) becomes

$$\eta = \int_{-s}^{\infty} f(x)dx = 1 - F(-s), \tag{13}$$

where F is the cumulative probability distribution, and $s = S/\sigma$, with S defined by (9), depends on h and p through their difference δ, that is

$$s = s(j, \delta; \eta) \equiv \delta + j\eta. \tag{14}$$

If the pdf has infinite support,

$$F(z) \equiv \mathcal{P}(x \leq z) = \int_{-\infty}^{z} f(x)dx. \tag{15}$$

In the case of a compact support $[x_m, \ x_M]$, one can also write:

$$\eta = 1 - F(-s) = \int_{\max\{x_m, -s\}}^{\max\{x_M, -s\}} f(x)dx. \tag{16}$$

Obviously, when $-s < x_m$, we have $\eta = 1$, and when $-s > x_M$ we have $\eta = 0$.

2.3 Hints from two extreme cases

Notice that in the absence of social influence, that is putting $j = 0$ in the above equations, the problem is simple because $s = h - p$ does not depend on η. Then, the fraction of buyers (13) is a monotonically increasing function of δ (equivalently, at fixed h, a decreasing function of the price p):

$$\eta = 1 - F(-\delta). \tag{17}$$

Another extreme case is the one of an homogeneous population: $H_i = H$ for every i - a situation obtained in the singular limit $\tilde{\sigma} = 1/j \to 0$ (the IWP distribution becoming a Dirac distribution). In that case every agent is faced to exactly the same decision problem, so that at equilibrium either $\eta = 0$ or $\eta = 1$. For each agent the surplus in case of adoption would be $H - P$ if no other agent adopt ($\eta = 0$), whereas if $\eta = 1$ the surplus is $H - P + J$. The solution $\eta = 0$ is valid if $H < P$, and the $\eta = 1$ solution is valid if $H > P - J$. Hence there is a domain, $P - J < H < P$, where the two solutions coexist. The whole population behaves as a single agent who either does not adopt, $\eta = 0$, or adopt, $\eta = 1$, with a different surplus whether he is 'in' or 'out of' the market: this is analogous to the problem of multi-equilibria with hysteresis in trade analyzed by Baldwin and Krugman [6], except that here the problem arises only at *the collective level*.

We have thus on one side, for $J = 0$ and σ finite, a unique well behaved equilibrium, and for $J > 0$ but $\sigma = 0$ a situation of multiequilibria. The question addressed in the following is thus to understand what happens 'in between'. We will see that when the social strength is large enough compared to the heterogeneity strength, the problem becomes more complex for the demand (and even more for the supply): beyond some critical value j_B of $j = J/\sigma$, equation (13) presents multiple solutions. The (possibly multiple) Nash equilibria are the solutions with economic meaning, i.e. those for which the demand decreases when prices increase.

The section 3 of the paper is devoted to a detailed study of the nature of the solutions of equations (13) and (14) with $j \geq 0$ for distributions satisfying very general smoothness hypotheses, detailed in the next section.

2.4 The idiosyncratic willingness-to-pay distribution

Since we are interested in the generic properties of the model, we explicit the general characteristics of the idiosyncratic willingness-to-pay (IWP) distribution covered by our analysis.

A meaningful pdf $f(x)$ (equation (1)) for the IWP must vanish in the limits $x \to \pm\infty$. For sufficiently regular pdfs, this can happen in two different ways: either the pdf decreases continuously to 0 as $x \to \pm\infty$, or it is strictly zero outside some compact support $[x_m, x_M]$. Most of the analysis in this paper is restricted to the class of pdfs obeying to the following hypotheses:

H1. *Modality*: f is *unimodal*, that is it has a *unique* maximum.

H2. *Smoothness*: f is non zero, continuous, and at least piecewise twice continuously differentiable inside its support, $]x_m, x_M[$, where x_m and x_M may be finite or equal to $\pm\infty$. Note that, since a pdf must be integrable, in the latter case f is stricly monotonically decreasing towards zero as $x \to \pm\infty$.

H3. *Boundedness*: the maximum of f, f_B (that may be reached at x_m or x_M if these numbers are finite), is *finite*:

$$f_B = \sup_x f(x) < \infty \tag{18}$$

Within the class of pdfs satisfying H1, H2 and H3, we will consider more specifically the important following prototypical cases:

1. *Unbounded supports*: The support of the distribution is the real axis; the pdf is continuous and twice continuously derivable on $]-\infty, \infty[$, with a unique maximum. A typical exemple, relevant to economics (see e.g. [1]), is given by the logit distribution, although more generally we do not assume that the pdf is symmetric. We make the following supplementary hypothesis, that amount to impose that the pdf decreases fast enough for $x \to \pm\infty$:

   H4. *Mean value*: the pdf has a finite mean value. Then, the smoothness condition H2 imposes that f decreases when $x \to \pm\infty$ faster than $|x|^{-1}$.

   H5. *Variance*: the pdf has a finite variance. Then, the smoothness condition H2 imposes that f decreases when $x \to \pm\infty$ faster than $|x|^{-2}$.

2. *Compact supports*: the support of the distribution is some interval $[x_m, x_M]$, with x_m and x_M finite; the pdf is continuous on $[x_m, x_M]$ and continuously derivable on $]x_m, x_M[$, with a unique maximum on $[x_m, x_M]$. Note that, since f has zero mean, $x_m < 0 < x_M$.

Hypothesis H2 and H3 exclude cases where the pdf is not a function but a distribution — containing, e.g., a Dirac delta —. Clearly, if the pdf's support is the real line, $]-\infty, +\infty[$, the boundedness hypothesis H3 is a consequence of the smoothness hypothesis H2. In the case of compact supports, H3 excludes pdfs diverging at a boundary of the support. Although hypothesis H3 is actually true under H2 if f is continuous on the closed interval $[x_m, x_M]$, we explicit it because some of our results are valid under H3 even for pdfs less regular than those satisfying H2. Hypothesis H5 is not necessary for the study of the aggregate demand, section 3. We explicit it here because it is used in the analysis of the supply side, section 4.

Generic results for unbounded support pdfs satisfying H1 to H5 are presented in the main body of the paper. They extend previous results obtained for a logistic distribution [66]. The analysis of other types of pdfs is left to Appendix B:

- In Appendix B.1 we present general results for bounded support pdfs. The case of a uniform distribution on a finite interval $[x_m, x_M]$, which corresponds to an interesting degenerate case (f is maximal at every point within the interval), has been presented elsewhere [43].
- In Appendix B.2 we extend the analysis to *fat-tail* distributions, which correspond to an important limiting case of pdfs with infinite variance (for such distributions, σ in (1) and in equations (10) and (11) is no longer the variance, but an arbitrary constant setting the units of H, J, P and C).
- Finally, in Appendix B.3 we extend the discussion to the case of multimodal pdfs (distributions with an arbitrary number of maxima): we derive the demand phase diagram for a generic smooth multimodal pdf; the case of a singular pdf is discussed on the particular example of a pdf with two Dirac peaks.

3 Aggregate choices and coordination dilemma

In this section we discuss the demand function, that is the relationship between price p and fraction of buyers (or adopters, in non market contexts) η, expressed by equation (13). As we have already seen, this means studying the relationship between η and the bare surplus $\delta = h - p$, and how it depends on the externality parameter j. We show that, for a large range of the parameters j and δ, the demand presents two equilibria which can be qualified as Nash equilibria from a game-theoretic point of view. This result is valid for any pdf satisfying the general hypothesis described in the preceding section.

3.1 The direct and inverse demand functions

The expected demand η^d at a given value of δ is obtained as the implicit solution of (13) and (14). As we will see, the application $\eta \to 1 - F(-s(\eta))$ may be a multiply valued function of η; it is thus preferable to express δ, or $p = h - \delta$, as a function of η, and determine the *inverse demand function* $p^d(\eta)$, that is, the price at which exactly $N\eta$ units of the good would be bought[3].

Under the hypothesis H2, the cumulative distribution F is a continuous and *strictly* monotonic function on $]x_m, x_M[$ and has a unique inflexion point. Hence it is invertible. Denoting Γ as the inverse of $1 - F(-s)$, we have the following equivalence:

$$\eta = 1 - F(-s) \iff s = \Gamma(\eta), \tag{19}$$

[3]In non market models, where generally $p = 0$, results in this section give the demand, or aggregate choice, $N\eta^d(j, h)$, that is, the relationship between the fraction of adopters, the average willingness to adopt of the population and the strength of the social interactions.

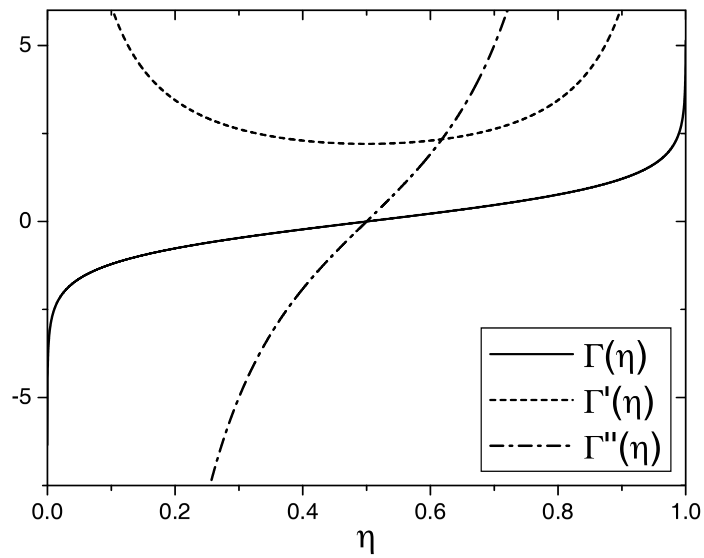

Figure 1: $\Gamma(\eta)$ *and derivatives as a function of* η *for the logistic pdf of unitary variance. Remark: all these functions diverge at* $\eta = 0$ *and at* $\eta = 1$.

with s defined by (14). For unbounded supports, $\Gamma(\eta)$ increases monotonically from $-\infty$ to $+\infty$ when η goes from 0 to 1 (see figure 1 for an example, and Appendix B for other cases). In the case of a compact support $[x_m, x_M]$, $\Gamma(\eta)$ takes the finite values, $\Gamma_0 \equiv \Gamma(0) = -x_M$, and $\Gamma_1 \equiv \Gamma(1) = -x_m$ for η respectively 0 and 1. Note that neither we assume f to be symmetric nor to have its maximum at $x = 0$.

Replacing s in the r.h.s. of (19) by its expression (14) we obtain

$$\delta = \mathcal{D}(j;\eta), \tag{20}$$

with

$$\mathcal{D}(j;\eta) \equiv \Gamma(\eta) - j\eta. \tag{21}$$

Note that $\mathcal{D}(j;\eta)$ depends on the parameter j but not on h. As we will see, for the analysis it is important to keep in mind the dependency of $\mathcal{D}$ on j. Actually, in section 3.2, we will have to consider $\mathcal{D}(j;\eta)$ as a function of the two variables, j and η. In the present subsection, we consider $\mathcal{D}(j;\eta)$ as a function of the single variable η, with j as a fixed parameter. Derivatives of $\mathcal{D}(j;\eta)$ with respect to η will therefore be denoted $\mathcal{D}'(j;\eta)$ unless otherwise stated.

Plots of $\mathcal{D}(j;\eta)$ against η for different values of j are presented on figure 2 for the logistic distribution. Solutions to (20) correspond to the intersections of these functions with horizontal lines at $y = \delta$.

The stable equilibrium values of the demand satisfy

$$\mathcal{D}'(j;\eta) \geq 0. \tag{22}$$

Thus, the intersections of $y = \delta$ with $\mathcal{D}$ when $\mathcal{D}' < 0$ correspond to unstable equilibria and will be ignored, as explained below.

From the definition of δ and (21), the inverse demand function is thus

$$p^d(\eta) = h - \mathcal{D}(j;\eta). \tag{23}$$

This function depends on both parameters h and j, and when necessary we will write $p^d(\eta) = p^d(h, j;\eta)$.

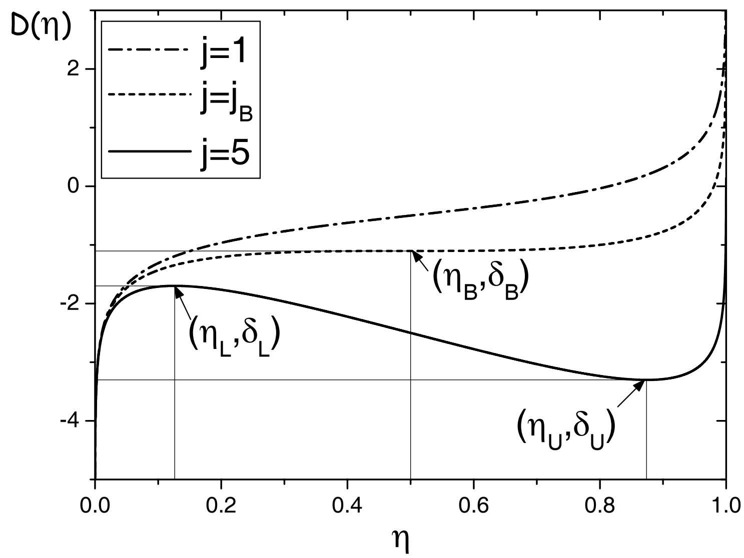

Figure 2: $\mathcal{D}(j;\eta)$ *as a function of* η *in the case of a logit distribution of the IWP, for three values of* j*:* $j < j_B$*,* $j = j_B$ *(*$j_B = 2.20532$ *for the logistic) and* $j > j_B$*.*

The solution $\eta = \eta^d(j,\delta)$ of equation (20) at a given value of δ gives the expected demand $N\eta^d(j,\delta)$ at a price $p = h - \delta$, for an externality parameter j. Hereafter we will drop the j and write $\eta^d(\delta)$, unless necessary.

As already mentioned, we will see that the demand $\eta^d(\delta)$ can be a multivalued function of δ for some range of parameters. On the contrary, since the function $\Gamma(\eta)$ is a uniquely defined function of η, so is $\mathcal{D}(j;\eta)$. For this reason, instead of considering (13), it is convenient to obtain the properties of the demand $\eta^d(\delta)$ from the analysis of equations (20) and (21).

Under assumption H2, $\Gamma(\eta)$ is at least piecewise three times continuously derivable on $]0,1[$; its derivative Γ' is continuous and strictly positive. In particular, at any point η in $]0,1[$, we have:

$$\mathcal{D}'(j;\eta) = \Gamma'(\eta) - j. \tag{24}$$

In the case of a compact support, the above equation also holds for the right and left derivatives at, respectively, $\eta = 0$ and $\eta = 1$.

In terms of the pdf f,

$$\Gamma'(\eta) = \frac{1}{f(-s)}\ , \text{ with } \ s = \Gamma(\eta). \tag{25}$$

Under H1, Γ' has a unique absolute minimum (qualitatively there is a unique point where the curvature of Γ changes from convex to concave; if Γ is smooth, it has a unique inflexion point). Thus

$$\min_{\eta} \Gamma'(\eta) = \frac{1}{f_B}\ > 0. \tag{26}$$

This minimum is reached at some value $\eta = \eta_B$:

$$\eta_B \equiv \arg\min_{\eta} \Gamma'(\eta). \tag{27}$$

If f is smooth enough at its maximum, then $\Gamma''(\eta_B) = 0$: η_B is the inflexion point of Γ. For symmetric pdfs, $\eta_B = 1/2$, but we do not restrict to this case.

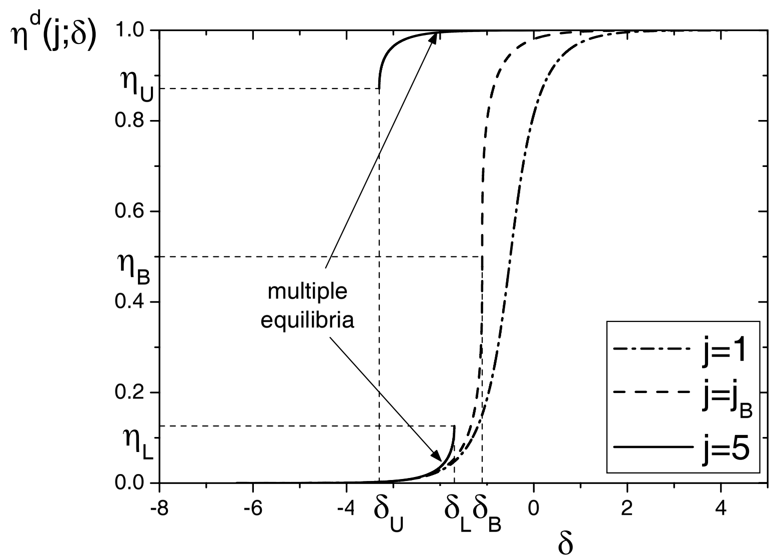

Figure 3: *Demand* $\eta^d(\delta)$ *as function of* $\delta \equiv h - p$ *in the case of a logit distribution of the IWP, for three values of* j*:* $j < j_B$, $j = j_B$ ($j_B = 2.20532$ *for the logistic) and* $j > j_B$*. Notice that the origin of the horizontal axis (*$\delta = 0$*) corresponds to* $h = p$*. Remark that prices* increase *from* right to left. *Unstable solutions, a curve of negative slope joining* δ_L, η_L *to* δ_U, η_U*, are not shown.*

As a consequence of the properties of $\Gamma(\eta)$, equation (24) gives that $\mathcal{D}'(j;\eta)$ is strictly positive for $j < j_B$, with

$$j_B \equiv \Gamma'(\eta_B) = \frac{1}{f_B}. \tag{28}$$

The value j_B separates two regions where the model presents qualitatively different behaviors. When $j < j_B$, the function $\mathcal{D}(j;\eta)$ is strictly increasing from $-\infty$ to $+\infty$ as η goes from 0 to 1. As a result it is invertible: for any δ in $]-\infty, +\infty[$, equation (20) has a unique solution $\eta^d(\delta)$.

If $j > j_B$, there is a range of values of δ for which (20) has several solutions. The existence of *multiple solutions* in the demand is thus a *generic property of discrete choice models with heterogeneous agents and social interactions* (externalities). This is true whatever the number of maxima of f, as shown in section B.3 of Appendix B. Actually, the domain where there is a unique solution, that is $0 \leq j \leq j_B = 1/f_B$, is very narrow if f_B is large: a leptokurtic distribution will have in general a narrower domain of unicity of the demand than a platykurtic distribution of same variance.

In our case of unimodal pdfs, equation (20) has three solutions for $j > j_B$, as illustrated on figure 2. The intermediate solution, laying on a branch with $\mathcal{D}'(j;\eta) < 0$, that is where η increases as $\delta = h - p$ decreases, is sometimes called a critical mass point in the literature [76]: it corresponds to a demand that would increase for increasing prices. Hence, in a *tatonnement* dynamics, it would correspond to an unstable solution separating the basins of attraction of the two stable equilibria. The marginal case, $j = j_B$, is a bifurcation point (hence the subscript B) where multiple solutions to (20) appear on increasing j. The stable equilibria of the demand that satisfy (22) are represented against δ on figure 3.

3.2 Demand phase diagram

3.2.1 The demand multiple-solution region

Let us consider more in details the behavior of the application $\delta \to \eta^d(\delta)$ in the case of a smooth unimodal pdf on $]-\infty, +\infty[$. Considerations specific to compact support pdfs are left to section B.1 of Appendix B.

The function $\Gamma(\eta)$ corresponding to pdfs satisfying H1 to H5 are at least three times continuously derivable on $]0, 1[$, and diverges towards $-\infty$ and $+\infty$ as η goes to 0 and 1 respectively. We have already seen that for $j < j_B$ there is a unique solution, and η^d goes from 0 to 1 as $\delta = h - p$ goes from $-\infty$ to $+\infty$.

For $j > j_B$, (20) has 3 solutions whenever $\delta_U < \delta < \delta_L$ (see figure 2), where δ_L and δ_U are the values of δ that satisfy the equality (marginal stability condition) in equation (22). That is, the boundaries of the multiple solutions region are the values for which $\mathcal{D}(j; \eta)$ has a horizontal slope (see figure 2):

$$\mathcal{D}'(j; \eta) = 0 \tag{29}$$

which is equivalent to

$$\frac{dp^d(\eta)}{d\eta} = 0. \tag{30}$$

Considering the definition (21) of $\mathcal{D}$, this means that on these boundaries $\mathcal{D}(j; \eta(j))$, *as a function of* j, is the Legendre transform of $\Gamma(\eta)$. Under our hypothesis H1, Γ' has a unique minimum, and necessarily tends towards $+\infty$ as η goes to either 0 or 1; Γ is strictly convex on $]\eta_B, 1[$, and strictly concave on $]0, \eta_B[$, hence the Legendre transform is well defined and unique on each one of these intervals: equation (29) for $j > j_B$ has indeed two solutions $\eta_L(j)$ and $\eta_U(j)$, given by

$$j = \Gamma'(\eta_\Lambda), \quad \Lambda = U, L. \tag{31}$$

with

$$\eta_L(j) < \eta_B < \eta_U(j). \tag{32}$$

From the knowledge of $\eta_U(j)$ and $\eta_L(j)$, using (20) one gets the marginal stability curves $\delta_U(j)$ and $\delta_L(j)$, that is, the extreme values of δ bounding the region where multiple solutions exist:

$$\delta_\Lambda(j) = \mathcal{D}(j; \eta_\Lambda) = \Gamma(\eta_\Lambda) - j\eta_\Lambda, \quad \Lambda = U, L. \tag{33}$$

As already stated, for $\delta_U < \delta < \delta_L$, equation (20) has three solutions. The curve $\eta^d(\delta)$ has two stable branches: an upper one with $\eta^d(j, \delta) > \eta_U(j) > \eta_B$, and a lower one with $\eta^d(j, \delta) < \eta_L(j) < \eta_B$; they are joined by a branch of unstable solutions — the above mentioned set of unstable equilibria (see figure 3) —. The upper branch has $\delta \geq \delta_U$, the lower one $\delta \leq \delta_L$. At the end points $\frac{d\eta^d}{d\delta}|_{L,U} = \infty$. In other words, solutions with large fractions of buyers, i.e. high η solutions, only exist for $\delta \geq \delta_U(j)$ whereas low η solutions exist only if $\delta \leq \delta_L(j)$. Since $\delta_U(j) \leq \delta_L(j)$, the system has multiple solutions for the demand η^d whenever $\delta_U(j) \leq \delta \leq \delta_L(j)$.

For $j = j_B$, these marginal stability curves merge at a single (degenerate) point $\delta_L(j_B) = \delta_U(j_B) = \delta_B$ with

$$\delta_B \equiv -\Gamma'(\eta_B)\, \eta_B + \Gamma(\eta_B) \tag{34}$$

This defines the bifurcation point B in the (j, δ) plane,

$$B \equiv \{j_B, \delta_B\}. \tag{35}$$

One should note that $\eta_{U,L}$ and $\delta_{U,L}$ depend on j (and on the function $\Gamma(.)$), but neither on h nor p.

In fact, the preceding analysis can be made more general because the main results may be obtained only based on the continuity and the convexity properties of Γ, without assuming any smoothness properties of the derivatives of f. Let us consider this alternative.

First, whatever the smoothness properties of f, the demand η^d must be a decreasing function of the prices: the economically acceptable values of the equilibrium demand, $\eta^d \in [0,1]$, have to increase when δ increases (p decreases). Thus, among the solutions of (20), the equilibria lie on the branches where $\mathcal{D}$ (defined by equation (21)), is an increasing function of η (for differentiable pdfs, this condition is given by equation (22)).

Next, let us analyze $\mathcal{D}(j;\eta)$ as a function of η (see figure 2). By continuity of the function $\Gamma(\eta)$, $\mathcal{D}(j;\eta)$ is a continuous function of $\eta \in [0,1]$. As $\eta \to 0$, $\mathcal{D} \to -\infty$, and as $\eta \to 1$, $\mathcal{D} \to +\infty$. Since Γ is concave on $]0,\eta_B]$, on increasing η from 0 within $[0,\eta_B]$, $\mathcal{D}(j;\eta)$ has a maximum, $\delta_L(j)$, on this interval. $\delta_L(j)$ is by definition the Legendre transform of Γ restricted to $]0,\eta_B]$. For $\eta \geq \eta_B$, Γ is convex, and thus $\mathcal{D}(j;\eta)$ has a minimum $\delta_U(j)$ on $[\eta_B,1]$, which is the Legendre transform of Γ restricted to $[\eta_B,1[$. Beyond this minimum, $\mathcal{D}(j;\eta)$ increases.

Now, for $j < j_B$, the minimum and the maximum of $\mathcal{D}(j;\eta)$ are both reached at η_B: $\mathcal{D}(j;\eta)$ increases monotonically as a function of η. Therefore, the solutions η^d to equation (20) are unique monotonically increasing functions of δ for each j. As a result, the inverse demand (23) is a uniquely defined continuously decreasing function of $\eta \in [0,1]$.

For $j > j_B$, the maximum $\delta_L(j)$ is reached at $\eta = \eta_L(j) \in]0,\eta_B[$. Beyond this maximum, $\mathcal{D}(j;\eta)$ decreases as η increases. The minimum $\delta_U(j)$ is reached at $\eta = \eta_U(j) \in]\eta_B,1[$: there is an intermediate interval $]\eta_L(j),\eta_U(j)[$ containing η_B where $\mathcal{D}(j;\eta)$ decreases with η, from δ_L to δ_U. No value of η within this interval can be a stable economic equilibrium. Hence, for δ ranging between these extrema of $\mathcal{D}(j;\eta)$ the demand $\eta^d(\delta)$ has two branches, a lower one for $\delta \leq \delta_L$, with $\eta^d(j,\delta) \leq \eta_L(j) < \eta_B$ and and an upper one for $\delta \geq \delta_U$, with $\eta^d(j,\delta) \geq \eta_U(j) > \eta_B$.

In the case of a continuously differentiable function, the preceeding results are recovered, since the Legendre transforms — the above mentioned minimum and maximum of $\mathcal{D}(j;\eta)$ for $j > j_B$ — are reached at the values of η solutions of (29). All this discussion based on convexity arguments can be extended to the case of multimodal pdfs, that is of distributions with more than one maximum. This is done in Appendix B.3.

3.2.2 The phase diagram

These results may be summarized on a *customers phase diagram* in the plane (j,δ), where we represent the boundary of the multiple solutions region, as in figure 4. These boundaries are the functions $\delta_\Lambda(j)$, $(\Lambda = L,U)$ defined by equations (33), which are the two branches of the Legendre transform of Γ, one for $\eta < \eta_B$ and the other for $\eta > \eta_B$. Note that in term of prices, the extreme values $\delta_L(j)$ and $\delta_U(j)$ correspond to prices $p_L(j,h) < p_U(j,h)$ given by

$$p_\Lambda(j,h) = h - \delta_\Lambda(j), \quad \Lambda = U, L. \tag{36}$$

By construction of the Legendre transforms, the branch $\delta = \delta_U(j)$ is concave, and the branch $\delta = \delta_L(j)$ is convex. In addition, under the smoothness hypotheses, along each branch of the multiple solutions region in the phase diagram:

$$\frac{d\delta_\Lambda(j)}{dj} = \frac{d\mathcal{D}(j;\eta_\Lambda(j))}{dj} = -\eta_\Lambda(j), \quad ; \quad \Lambda \in \{L,U\} \tag{37}$$

— a property which is easily checked by deriving (33) with respect to j and making use of (31) —. This means that the tangent to the boundary has a slope given by the value of η of the solution which is marginally stable on this boundary (the solution which appears/disappears as one crosses the boundary). This properties shows in particular that the δ width of the multiple solutions region increases with j as a results of the convexity properties of the functions $\delta_\Lambda(j)$,

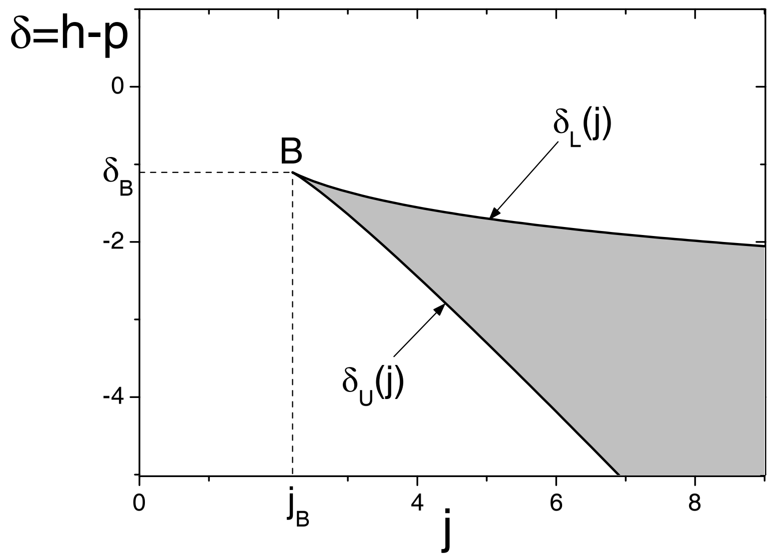

Figure 4: *Demand phase diagram on the plane* ($j = J/\sigma,\ \delta = (H-P)/\sigma$)*, for a smooth IWP distribution (here the logit). In the shaded region the demand presents multiple Nash equilibria. Outside this region, the demand is a single valued function of* j *and* δ

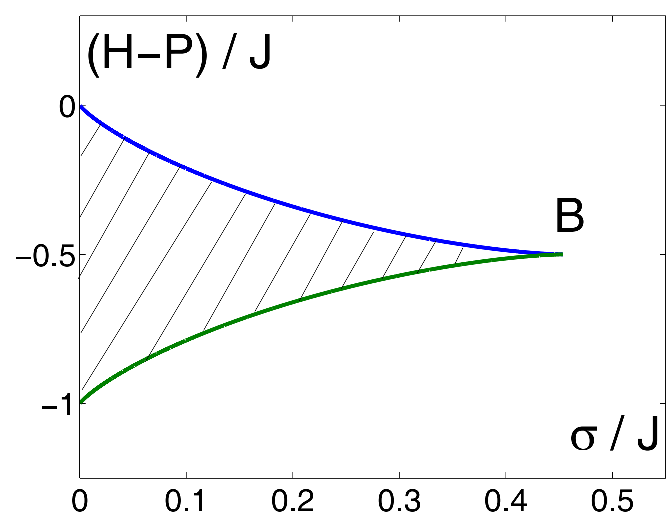

Figure 5: *Demand phase diagram in the plane* ($\tilde{\sigma} = \sigma/J,\ \tilde{\delta} = (H-P)/J$)*, for a smooth IWP distribution (here the logit). Inside the dashed region the demand presents multiple Nash equilibria. Outside this region, the demand is a single valued function of* $\tilde{\sigma}$ *and* $\tilde{\delta}$.

($\Lambda = L, U$). This may also be seen from (32), since the slope of the L boundary (the one corresponding to η_L) is larger than that of the U boundary (defined through η_U). At the bifurcation point B, these two boundaries merge, and, according to (37), have a common slope $-\eta_B$.

Referring to figure 2, upon increasing δ from $-\infty$, we have the following picture: if $j < j_B$, the fraction η increases smoothly from 0 and reaches its upper value 1 for $\delta \to \infty$. That is, to each value of the bare surplus δ, — or each value of the average willingness to adopt, in non-market situations — corresponds a unique fraction of buyers/adopters. In the phase diagram, figure 4, these solutions lie on the white region. On the other hand, if $j > j_B$, when δ reaches the value $\delta_U(j)$, a second, high η solution appear besides the low η one. These solutions co-exist, and the low η solution disappears when δ increases beyond $\delta_L(j)$, when only the high η solution survives. The parameter values for which there are multiple equilibria is the grey region of the phase diagram, figure 4. Notice that in this region, there exists a third solution that we neglected because it corresponds to the unstable case where the demand would increase with the price (or decrease with the bare surplus).

As mentioned in section 2.2, it is also useful to consider the same results in term of the parameters ($\tilde{\sigma} = \sigma/J$ and $\tilde{\delta} = (H-P)/J$) (see (12). The phase diagram in the plane $(\tilde{\sigma}, \tilde{\delta})$ is shown on figure 5. For large heterogeneity (σ/J larger than $\tilde{\sigma}_B \equiv 1/j_B$), there is a single smooth solution. For weak enough heterogeneity ($\tilde{\sigma}_B < 1/j_B$), there is a domain with multiple solution. In the limit $\tilde{\sigma} \to 0$, one recovers the simple results for an homogeneous population, as briefly discussed section 2.3.

3.2.3 Vicinity of the bifurcation point

The vicinity of the bifurcation point B in the phase diagram is of particular interest. Under the smoothness assumption H2, we can study analytically its properties. Let us consider $j = j_B + \epsilon$ with $0 < \epsilon << 1$. Expanding (31) about j_B, remembering that $\Gamma''(\eta_B) = 0$, one gets, to the lowest order in ϵ, the singular behavior

$$\eta^d_{L,U} = \eta_B \pm \sqrt{\frac{2}{\Gamma'''(\eta_B)}}\ \epsilon^{1/2}, \tag{38a}$$

$$\delta_{L,U}(j) = \Gamma(\eta_B) - \eta_B\ j_B - \eta_B \epsilon \mp \frac{2}{3}\sqrt{\frac{2}{\Gamma'''(\eta_B)}}\ \epsilon^{3/2}. \tag{38b}$$

The above singular behavior are typical examples of scaling properties which are *universal*: the same scaling is obtained for any smooth distribution. From studies in statistical physics one expects the exponents (e.g., here, 1/2 for the behavior of η) to depend mainly on the structure of the network of interactions: the exponents would be different at the analogous critical point for the model with agents situated on the vertices of a d-dimensional square lattice and interacting only with their nearest neighbors. Typically the exponents to depend on d up to some critical dimension d_c, above which the exponents become identical to the 'mean-field' exponents, those obtained with the global neighborhood. For the present model, other universal scaling properties have been obtained, in relation with the hysteresis effects [79], and these have been used in order to analyze empirical socio-economics data [65].

In section 4, when considering the supply side, we will see that the bifurcation point $B = \{j_B, \delta_B\}$ in the (j, δ) plane gives a singular point $\{j_B, h_B \equiv \delta_B\}$ in the (j, h) plane, to be still denoted by B, which plays an important role in the phase diagram associated to the optimal strategy for the monopolist.

3.2.4 Pareto optimality and coordination

Each one of the equilibria $\eta^d(j,\delta)$ discussed in the preceding section is a Nash equilibrium for the customers, at a posted price p. In this section we show that, whenever multiple solutions exist, that is for $j > j_B$, the solution with the largest η is Pareto optimal. This is the solution $\eta^d(j,\delta)$ that satisfies $\eta^d(j,\delta) \geq \eta_U(j)$.

Let us recall that if a customer i decides to buy, it is because his (normalized) surplus

$$s_i = \delta + j\eta + x_i, \tag{39}$$

is positive. When $s_i < 0$ he doesn't buy and his surplus vanishes. Thus, his actual surplus is $w_i = s_i\omega_i$ (see section 2). Consider now the two equilibria $\eta^d(j,\delta)$ in the region $\delta_U(j) < \delta < \delta_L(j)$ (see the curve $\mathcal{D}(j;\eta)$ for $j = 5$ on figure 2). Let's call upper (lower) equilibrium the solutions $\eta^d(j,\delta) \geq \eta_U(j)$ ($\eta^d(j,\delta) \leq \eta_L(j)$). In either equilibrium, the agents who buy are those with $x_i > -\delta - j\eta^d(j,\delta)$. Since $\eta^d_L(j) < \eta^d_U(j)$, agents with $x_i < -\delta - j\eta^d_U(j)$ are not buyers in any one of the equilibria; agents with $x_i > -\delta - j\eta^d_L(j)$ are buyers in both equilibria. Those with $-\delta - j\eta^d_U(j) < x_i < -\delta - j\eta^d_L(j)$ are buyers only in the upper equilibrium, and their utility is thus larger (strictly positive instead of zero) in that case. Moreover, even those agents that would buy in both cases have a larger surplus if the realized equilibrium is the upper one. Hence, in the upper equilibrium all these agents have a larger surplus than in the lower one. This situation with two possible Nash equilibria, where the strictly dominant one is risk dominated, belongs to the class of coordination problems in game theory.

Whether a Nash equilibrium — and which one in the case of multiple equilibria — is actually realized depends on the rationality of the agents and the information they have access to. In the context of bounded rationality and of repeated choices, a natural hypothesis is that agents estimate what will be the fraction of adopters, and may base their estimate on previous observations. In this paper we will not discuss these issues, that we are currently analyzing. Some partial results (dynamics with myopic agents and with various reinforcement learning paradigms) are discussed elsewhere [43, 78].

3.3 Summary of the generic customers' model

To summarize this section, if the social influence is small enough to satisfy the condition $j < j_B$, at each value of the gap $\delta = h - p$ between average IWP and price, there is a unique solution $\eta^d(\delta)$ of equation (20). This demand is a monotonic increasing function of δ. However, if the social influence is large enough ($j > j_B$), there is a range of values $\delta_U(j) < \delta < \delta_L(j)$ for which two different (stable) solutions exist, a high demand one ($\eta^d(j,\delta) \geq \eta_U(j) > \eta_B$) and a low demand one ($\eta^d(j,\delta) \leq \eta_L(j) < \eta_B$). In this region, the customers are faced with a coordination problem. If δ is modified dynamically within this range, the demand may jump abruptly between these two solutions, a situation analogous to so called *first order phase transitions* in physics. Outside the range $[\delta_U(j), \delta_L(j)]$, there is a single solution, like in the low j case.

In market contexts (and in particular for the market analysis of the next section), it is useful to consider the inverse demand p^d instead of δ, as in standard approaches. In figure 6 we plot the values of $p - h$ as a function of the demand η and the strength j of the social externality for the case of a logistic distribution.

Closely related or formally equivalent models have been studied in various contexts, for specific probability distributions. In particular, as mentioned in section 1.2, Durlauf and co-workers[30, 31, 19, 20] have studied the particular case of a logit pdf, and showed that for j larger than a threshold value there are multiple coordination equilibria. Here we have shown that such coordination equilibria are *generic*: they arise whenever the IWP distribution has a maximum. We showed that the threshold j_B, which corresponds to the onset of a bifurcation in

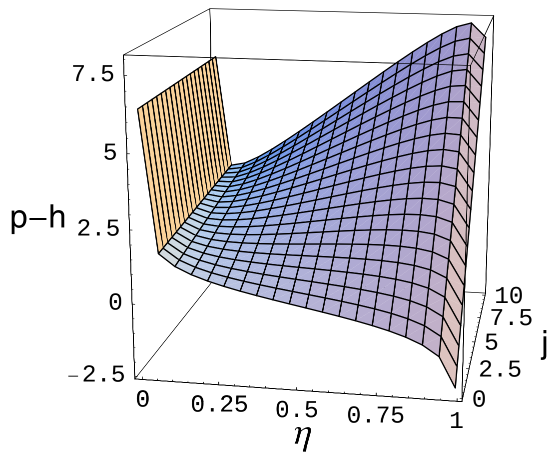

Figure 6: *Inverse demand $p - h \ \ (= -\delta)$ as a function of η for different externality strength values j, illustrated on the case of a logistic IWP distribution.*

the customers phase diagram, is determined by the maximum f_B of the IWP pdf: $j_B = 1/f_B$. Although most of the analysis has been done for smooth pdfs, we have shown that the generic behavior stems only from convexity properties of the function $\Gamma(\eta)$, that is from the fact that the pdf $f(x)$ is strictly increasing for $x < x_B$ and strictly decreasing for $x > x_B$, where x_B is the mode of f.

Some specific properties which arise for distributions with compact support will be discussed in section B.1, and the case of distributions with infinite variance will be studied in section B.2. In Appendix B.3 we extend the results of this section to multimodal distributions.

4 Supply side: the monopolist's dilemma

4.1 Monopolist model

In this section we examine the consequences of the social interactions among customers on the simplest market considered in economics textbooks: that of a monopolist who has to determine the price P and the number $N\eta$ of units of the good to put in the market in order to maximize his profit $N\Pi$ with

$$\Pi = (P - C)\,\eta. \tag{40}$$

C is the production cost per unit. Hereafter we make the simplifying hypothesis that C is independent of the total number of produced units, to keep the model simple. This hypothesis corresponds to a widely used assumption in networks economics [49, 55]. One can easily find many examples of constant marginal cost, and in some cases quasi-null marginal cost, like the cost of replicating software, musical or audiovisual digital files. Such assumption was criticized by Leibowitz and Margolis [56, 57], who argue that marginal costs are increasing functions of the produced quantities. This issue will however not be addressed here because our purpose is mainly to exhibit the simplest non trivial market consequences of the customers model of the previous section. Including increasing marginal costs would require a different analysis, because considering a total cost $N\eta C$ increasing faster than linearly with the population size N needs focusing on finite size effects.

In the analysis of the customers problem, we have seen that P appears only through $P - H$, which we can write as $(P - C) - (H - C)$. Hence, from the point of view of the monopolist, the relevant normalized variables are

$$j \equiv \frac{J}{\sigma}, \quad h \equiv \frac{H - C}{\sigma}, \quad p \equiv \frac{P - C}{\sigma}, \quad c \equiv \frac{C}{\sigma}. \tag{41}$$

This does not introduce any change in the analysis of the customers problem, since $\delta = h - p$ is the same with the two normalizations (10) and (41), but it simplifies the analysis of the monopolist optimization problem. Otherwise stated, without loss of generality, one can assume $c = 0$.

The monopolist has to choose the (normalized) price p and the number $N\eta$ of units of the good to put on the market in order to maximize his profit

$$\pi = p\eta, \tag{42}$$

knowing the demand function discussed in the previous section.

For $j = 0$, the optimization of the monopolist's profit is straightforward. The optimal price is solution of $\partial\pi/\partial p = 0$. Using (17) and the definition of δ (11), it gives

$$\frac{f(-\delta)}{1 - F(-\delta)} \;=\; \frac{1}{p}, \tag{43}$$

Setting $x \equiv -\delta \equiv p - h$, the solution x to this equation satisfies

$$\frac{1 - F(x)}{f(x)} - x = h, \tag{44}$$

which only depends on the details of the distribution $f(x)$. A solution of (44) is indeed a maximum if $\partial^2\pi/\partial p^2 < 0$, which, making use of the first order condition (44), can be written as

$$2f^2(x) \;+\; (1 - F(x))\, f'(x) \;>\; 0. \tag{45}$$

Even in this simple case, it remains to see whether equation (44) has one or several solutions.

In this section we show that the social interactions have two consequences: one of them is that the optimal profit has two maxima when the social interaction strength is larger than a threshold $j_A \leq j_B$. Thus, even if the demand is a monotonic function of the price, the monopolist has to make the correct strategic choice and post the price corresponding to the maximum maximorum of the profit. A second, even more drastic consequence, is that for $j > j_B$ the coordination problem of the customers makes it difficult — if not impossible — for the monopolist to make the correct strategic choice. Within this region of parameters, since the customers have two Nash equilibria, the price alone is not sufficient to determine the customers' decisions. The rest of the paper is devoted to a detailed study of the maximization of the profit for different distributions satisfying the general hypothesis presented in section 2.4.

4.2 Optimal price and (effective) supply functions

Formally, given the customers model of the previous section, the monopolist has thus to maximize the expected profit (per customer)

$$\pi(\eta, p) = p\,\eta \tag{46}$$

under the condition that the demand η and the price p are related through (20).

There are different ways of analyzing this maximization: one may consider that η is a function of p, $\eta = \eta^d(p)$ (as done in [66] for the particular case of the logit pdf), and impose the condition (23) using a Lagrange multiplier; or that p is a function of η, $p = p^d(\eta)$, and impose the condition $\eta = \eta^d(p)$ using a Lagrange multiplier. In section A.1 of Appendix A we propose a reasoning that leads (obviously) to the same results, but deals symmetrically with the variables p and η, and puts forward the analogy between the demand and the supply equations.

We find that the optimal price is the solution of the following equations:

$$p = p^s(\eta), \tag{47a}$$

$$p = p^d(\eta), \tag{47b}$$

where

$$p^s(\eta) = -\eta \frac{dp^d(\eta)}{d\eta}, \tag{48}$$

and $p^d(\eta)$ is given by equation (23), that we rewrite here,

$$p^d(\eta) = h - \mathcal{D}(j;\eta) = h + j\eta - \Gamma(\eta). \tag{49}$$

$p^s(\eta)$ can be seen as an effective *inverse supply function*. Note that a positive supply price, $p^s(\eta) > 0$, is equivalent to having an inverse demand decreasing with η, which is the consistent solution of the demand side for an economic system. However, $p^s(\eta)$ is not a true supply function, since it results from the monopolist's optimization program, itself based on the knowledge of the demand function. Nevertheless, it has all the properties of an inverse supply function, and the analysis of the equilibrium can be understood from the equality between demand and supply, $p^s(\eta) = p^d(\eta)$. Taking (48) into account, this equality takes the simple form

$$\frac{d\,(\eta p^d(\eta))}{d\eta} = 0. \tag{50}$$

From the expression (49) of p^d one has

$$p^s(\eta) = \eta[\Gamma'(\eta) - j] = \eta \mathcal{D}'(j;\eta). \tag{51}$$

Introducing (48) into the second order condition (A-7) we obtain:

$$\frac{d}{d\eta}[p^d(\eta) - p^s(\eta)] \leq 0. \tag{52}$$

This is a natural condition: it means that as the demand increases, the difference between the inverse demand and supply functions should decrease. For $\eta \neq 0$, this condition can also be written as $-\frac{1}{\eta}\frac{d(\eta p^s(\eta))}{d\eta} \leq 0$, which gives the condition of maximum (A-7) under another interesting form,

$$\frac{d(\eta p^s(\eta))}{d\eta} > 0. \tag{53}$$

Notice also that, introducing (48) into (52) we obtain the stability condition under a more familiar form,

$$d^2(\eta p^d(\eta))/d^2\eta \leq 0. \tag{54}$$

Making use of (51) we can explicit the second order condition in terms of Γ:

$$\frac{1}{2}\eta\Gamma''(\eta) + \Gamma'(\eta) - j \geq 0. \tag{55}$$

Using the first order conditions (47) and the definition of p^d given by equation (49), (55) may also be written as follows:

$$2p + \eta^2\Gamma''(\eta) \geq 0, \tag{56}$$

Under the smoothness hypothesis H2, we have seen that $\Gamma''(\eta)$ changes its sign only once, at $\eta = \eta_B$ where $\Gamma'' = 0$, so that $\Gamma''(\eta) > 0$ for $\eta > \eta_B$. As a result, one can already state that any solution of the equilibrium conditions (47a) and (47b) for which $p = p^d(\eta) > 0$ and $\eta > \eta_B$ is a (possibly local) maximum of the profit.

In order to obtain the monopolist's optimal price, one has to determine the value of η satisfying equations (47a) and (47b), that is $p = p^s(\eta) = p^d(\eta)$. Then, inserting the obtained value on either (47a) or (47b) gives the price that optimizes the profit. Using equations (49) and (51) we can write (47a) and (47b) as

$$h = \frac{d}{d\eta}[\eta\, \mathcal{D}(j;\eta)] \ \equiv \widehat{\mathcal{D}}(j;\eta). \tag{57}$$

$\widehat{\mathcal{D}}(j;\eta)$ is the average willingness to pay of a population where a fraction η of individuals are buyers at the supply-demand equilibrium.

Let us define $\widehat{\Gamma}(\eta)$ by

$$\widehat{\Gamma}(\eta) \equiv \frac{d[\eta\Gamma(\eta)]}{d\eta} = \Gamma(\eta) + \eta\Gamma'(\eta). \tag{58}$$

so that, taking (21) into account:

$$\widehat{\mathcal{D}}(j;\eta) = \frac{d[\eta\Gamma]}{d\eta} - 2j\eta = \widehat{\Gamma}(\eta) - \hat{\jmath}\eta = \mathcal{D}(j;\eta) + \eta\mathcal{D}'(j;\eta). \tag{59}$$

These definitions show that equation (57) has the very same structure as equation (20) of the demand, with $\widehat{\Gamma}$ and $\widehat{\mathcal{D}}$ being respectively the analogues of Γ and $\mathcal{D}$, and with $\hat{\jmath} \equiv 2j$ instead of j. Moreover, the maximum condition (55) is (using the same convention for the derivatives as in the preceding sections)

$$\widehat{\mathcal{D}}'(j;\eta) \geq 0. \tag{60}$$

Given the values h and j characterizing the customers, the monopolist's optimal solution is obtained as follows: first, the optimal fraction of buyers from the monopolist's point of view,

$\eta^s(j,h)$ is obtained through solving equation (57). This allows to determine the optimal price using (48):

$$p^s(j,h) = \eta^s \mathcal{D}'(j;\eta^s) = \widehat{\mathcal{D}}(j;\eta^s) - \mathcal{D}(j;\eta^s), \tag{61}$$

where the last equality stems from (59). Remark that the first equality shows that stability of the customers' solution guarantees that the optimal price is positive. Finally, the optimal profit is

$$\pi^s(j,h) = \eta^s(j,h)\, p^s(j,h).$$

If there is more than one solution, the one corresponding to the highest profit has to be selected among those that satisfy (60). The properties of these solutions depend on the characteristics of the function $\widehat{\mathcal{D}}(j;\eta)$, whose analysis follows the same lines as that of $\mathcal{D}(j;\eta)$ in section 3. By analogy with the customers model, we can conclude that, *if* the function $\widehat{\Gamma}$ is monotonically increasing with η, like Γ, then depending on the value of $\hat{\jmath}$, equation (57) will present either a single solution or multiple solutions for certain range of the average IWP h. In the latter case, the monopolist has to choose the one that maximizes his profit which, under the smoothness hypothesis H2, must satisfy (55).

In section A.2 of Appendix A we study the conditions that must satisfy a (smooth) pdf f for $\widehat{\Gamma}$ to be a monotonic, strictly increasing function of η for all $0 < \eta < 1$, that is such that

$$\widehat{\Gamma}'(\eta) > 0 \quad \forall \quad 0 < \eta < 1. \tag{62}$$

This condition, expressed in terms of the pdf $f(x)$ and its cumulative $F(x)$, gives equation (45), which may be rewritten as

$$\frac{d^2}{dx^2}\,\frac{1}{1-F(x)} \;>\; 0, \tag{63}$$

meaning that the function $1/\int_x^\infty f(u)du$ is convex. Considering the equation for the demand (13), this condition is equivalent to state that, in the absence of externalities ($j = 0$), $1/\eta(p)$ is a convex function of p:

$$\frac{d^2}{dp^2}\,\frac{1}{\eta(p)} \;>\; 0. \tag{64}$$

This is reminiscent of a convexity condition generally assumed in economics, namely

$$\frac{d^2}{dp^2}\,\log\frac{1}{\eta(p)} \;>\; 0. \tag{65}$$

However (64) is weaker than (65) — actually (65) implies (64) —. Hence, limiting our analysis to pdfs that satisfy (64) is not a big restriction, since anyway this class is larger than the one usually considered in economics.

In the following we concentrate on the generic case of pdfs such that $\widehat{\Gamma}(\eta)$ is monotonically increasing, with a single inflexion point at η_A. Then the analysis, summarized in the following section, follows the same lines as for the customers model.

4.3 Supply phase diagram

4.3.1 The supply multiple-solutions region

The analysis of the profit optimization in the case of functions $\widehat{\Gamma}$ that increase monotonically from $-\infty$ to $+\infty$ when η goes from 0 to 1 follows the same lines as the analysis of the customers system, section 3. In particular, there is a point A that plays for the supply the same role as

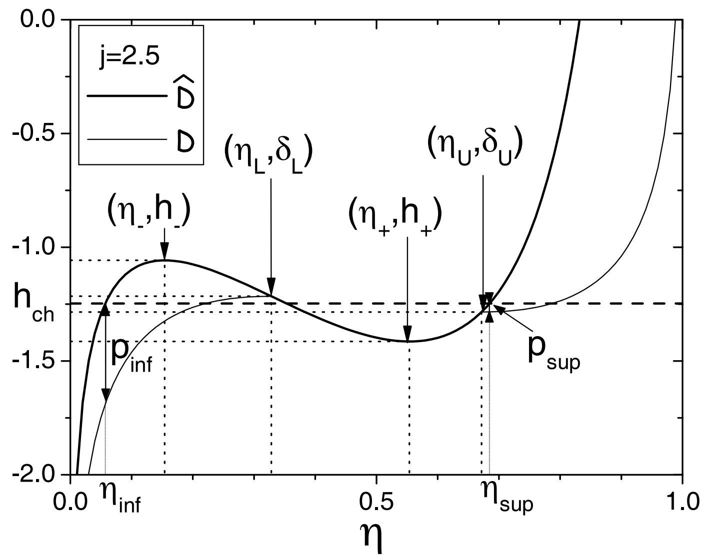

Figure 7: *Functions $\mathcal{D}(j;\eta)$ and $\widehat{\mathcal{D}}(j;\eta)$ showing the different boundaries of the monopolist's phase diagram parameterized by η. These curves, drawn here for a logistic distribution and $j > j_B$ (for the logistic, $j_B = 2.20532$), are typical of any smooth pdf. h_c is the value of h at which the monopolist should change his strategy, provided that the customers coordinate themselves on the fixed point expected by the monopolist.*

the point B for the demand. Figure 7 shows $\mathcal{D}(j;\eta)$ and $\widehat{\mathcal{D}}(j;\eta)$ for $j = 2.5 > j_B$ in the case of a logit distribution. The bifurcation point A corresponds to η_A given by

$$\eta_A \equiv \arg\min_{\eta} \widehat{\Gamma}'(\eta). \tag{66}$$

A straightforward consequence of inequalities (A-9) is that:

$$\eta_A \leq \eta_B, \quad \text{and} \quad \Gamma'(\eta_A) \leq 2\Gamma'(\eta_B). \tag{67}$$

If f is smooth enough, we can explicit the second derivative of $\widehat{\Gamma}$:

$$\widehat{\Gamma}''(\eta) = 3\Gamma''(\eta) + \eta\Gamma'''(\eta), \tag{68}$$

and η_A satisfies $\widehat{\Gamma}''(\eta_A) = 0$. Since $\Gamma'''(\eta) > 0$ for all η, from (68) we have consistently $\eta_A \leq \eta_B$.

Taking (59) into account, the condition that the extremum of the profit be a maximum, equation (60), is equivalent to

$$\widehat{\Gamma}'(\eta) \geq \hat{j} = 2j. \tag{69}$$

Like in the demand model, this defines a value $j_A = \widehat{\Gamma}'(\eta_A)/2$ at which there is a bifurcation in the profit's behaviour. For $j < j_A$, equation (57) has a single solution whatever the value of h. For $j \geq j_A$, there are three intersections of h with $\widehat{\mathcal{D}}(j;\eta)$. The solution corresponding to the intermediate value of η has $\widehat{\mathcal{D}}'(j;\eta) < 0$; it corresponds thus to a *minimum* of the profit. The extreme solutions (smallest and largest η) correspond to relative maxima of the profit, and the profits of both solutions must be compared in order to select the maximum maximorum.

The multiple extrema region is bounded by the values of η that satisfy $\widehat{\mathcal{D}}'(j;\eta) = 0$, that is, by the Legendre transforms of $\widehat{\Gamma}$. We denote them '+' and '-', with $\eta_- < \eta_A < \eta_+$ (η_A, by (66), is the inflexion point of $\widehat{\Gamma}(\eta)$). Their values are determined following the same steps as for the customer's model, and have the same form as equations (31) and (33), but with $\widehat{\Gamma}(\eta)$, $\hat{\jmath}$ and h instead of $\Gamma(\eta)$, j and δ respectively. In terms of $\Gamma(\eta)$ and its derivatives, the values of $\eta_+(j)$ and $\eta_-(j)$ are obtained through the inversion of:

$$j = \Gamma'(\eta_k) + \eta_k \frac{\Gamma''(\eta_k)}{2}, \quad k \in \{+,-\}. \tag{70}$$

The extreme values of h for which there exist multiple solutions for the supply are:

$$h_k(j) = \widehat{\Gamma}(\eta_k) - \hat{\jmath}\eta_k = \Gamma(\eta_k) - \eta_k \Gamma'(\eta_k) - \eta_k^2 \Gamma''(\eta_k), \quad k \in \{+,-\}. \tag{71}$$

These are the boundaries of the multiple extrema region of the supply phase diagram in the plane (j,h), discussed below (see fig. 8 for an example). They merge at η_A, which satisfies simultaneously $\widehat{\mathcal{D}}'(j;\eta) = 0$, and $\widehat{\mathcal{D}}(j;\eta)'' = 0$. This defines the point A through:

$$\begin{cases} A \equiv \{j_A \equiv \widehat{\Gamma}'(\eta_A)/2, \ h_A \equiv \widehat{\Gamma}(\eta_A) - \eta_A \widehat{\Gamma}'(\eta_A)\}, \\ \eta_A \leq \eta_B \ ; \ j_A \leq j_B \end{cases} \tag{72}$$

where the inequality between j_A and j_B stems from equation (71).

These results may be also restated as follows: in the plane (j,h), the boundary of the region of multiple solutions is given by the set of marginal stability points ($\widehat{\mathcal{D}}'(j;\eta) = 0$), which can be seen as a curve $(j = j^s(\eta), h = h^s(\eta))$ parameterized by η:

$$\begin{cases} j^s(\eta) = \Gamma'(\eta) + \eta \frac{\Gamma''(\eta)}{2} \\ h^s(\eta) = \Gamma(\eta) - \eta\Gamma'(\eta) - \eta^2 \Gamma''(\eta) \\ 0 \leq \eta \leq 1. \end{cases} \tag{73}$$

This boundary has two branches, a lower branch corresponding to $\eta_A < \eta < 1$, and an upper branch corresponding to $0 < \eta < \eta_A$. These two branches merge at the singular point A. If one crosses these lines coming from low h to high h, or from low j to high j, the lower branch is the one at which the $'+'$ solution appears (solution with a fraction of customers larger than η_A), and the upper branch is the one on which the $'-'$ solution disappears (the solution with a number of customers smaller than η_A).

Like in the customer's problem, from the construction of the Legendre transform, the slope of each branch is given by the value of η:

$$\frac{dh^s}{dj^s} = -2\eta. \tag{74}$$

This is easily checked by computing $dj^s(\eta)/d\eta$ and $dh^s(\eta)/d\eta$ from (73). In particular at point A, the tangent is the same for $\eta \to \eta_A$ from above and from below, that is, the point A is an apex. One can note, like in the customers problem, that the width of the region with multiple extrema increases with j because $\eta_- < \eta_+$.

4.3.2 Null price boundary

The acceptable solutions at a given h are those having $\widehat{\mathcal{D}}(j;\eta) \geq \mathcal{D}(j;\eta)$, which guarantee positive prices. However, the condition of maximum profit (56) may give negative solutions for p in the region where $\Gamma'' > 0$, that is, on the $\eta \geq \eta_+$-fold: in order to sell to a maximum of customers (solution '+'), the seller would have to lower the price below his cost. This negative price solution nevertheless is a local maximum of the profit: in fact, it is the price that minimizes the monopolist's deficit when the fraction of customers lies in the high fraction of buyers branch. Such economically unacceptable solutions arise for the values of η that satisfy simultaneously the inequalities $\mathcal{D}'(j;\eta) \leq 0$ and $\widehat{\mathcal{D}}'(j;\eta) \geq 0$, in other words, for $\eta\Gamma'' \geq -2(\Gamma' - j) \geq 0$. Thus, these solutions exist only for $\mathcal{D}'(j;\eta) \leq 0$, corresponding to the customers unstable fixed points. Clearly, the customer's bifurcation point B satisfies these conditions for the equality, and consistently, the above equations give $j = j_B$, $h = \delta_B$ for $\eta = \eta_B$. This defines a corresponding point B for the monopolist's phase diagram in the plane (j, h):

$$B \equiv \{j_B, h_B \equiv \delta_B\}. \tag{75}$$

This point B thus belongs to the lower branch of the boundary of the multiple solution region. The $p = 0$ line starts on this lower marginal stability curve $h_+(j)$ (see equation (71)) at point B and extends on the $j > j_B$ side.

Therefore, in the (j, h) plane the domain of existence of the solution corresponding to the optimal profit with a large value of η is bounded below by $h_+(j)$ between points A and B, and for $j > j_B$ by the null price line which is given by the customers bound for the U solution, $h(j) = \delta_U(j)$ (that gives $h(j)$ for $p = 0$).

Parameterized by η, the (half) line of null price is given by

$$\text{for } \; \eta_B \leq \eta \leq 1 : \begin{cases} j_0 & = \Gamma'(\eta) \\ h_0 & = \Gamma(\eta) - \eta\Gamma'(\eta). \end{cases} \tag{76}$$

and starts at the bifurcation point B of coordinates $(j_B, h_B = \delta_B)$.

4.3.3 The phase diagram

The results of the previous sections can be summarized on a monopolist's phase diagram, where the regions corresponding to different solutions are represented in a plane of axis (j, h), as on figure 8. Referring back to figure 7, they may be easily understood in terms of h, the average willingness to pay of the customers population.

When $j < j_A$, both curves $\mathcal{D}$ and $\hat{\mathcal{D}}$ are monotonically increasing functions of η. Thus, whatever the value of h, the intersection of the line $y = h$ with $\hat{\mathcal{D}}$ determines the fraction of buyers $\eta(h)$ (the abscissa of the intersection point), and the difference $\hat{\mathcal{D}}(\eta(h)) - \mathcal{D}(\eta(h))$ is the monopolist's optimal price.

For $j_A < j < j_B$, $\mathcal{D}(\eta)$ is still monotonically increasing but not $\hat{\mathcal{D}}$. The latter has a decreasing slope for $\eta_- < \eta < \eta_+$ where $\eta_\pm(j)$ are the solutions of equation (70). There is a range of values of h giving raise to two local maxima of the profit, corresponding to two different strategies for the monopolist: either to attract a large fraction of buyers at low prices or to look for few buyers willing to pay high prices. The optimal strategy (the maximum maximorum of the profit) switches from one to the other at a critical value h_c. Since within this range of j the fraction of customers is a unique function of η, the monopolist posted price drives the customers to the expected equilibrium.

When $j > j_B$ both $\mathcal{D}$ and $\hat{\mathcal{D}}$ present a negative slope, but for different ranges of η. As a consequence of the properties of $\hat{\Gamma}$, the monopolist's branch of small ηs — corresponding to intersections of $y = h$ with $\hat{\mathcal{D}}$ (that is $\eta < \eta_-(j)$, see figure 7) — corresponds to positive prices, since the corresponding inverse demand $\mathcal{D}$ is an increasing function of η and lies below $\hat{\mathcal{D}}$. On the contrary, for the monopolist's high ηs branch, with $\eta > \eta_+$ and $h > h_+$, their exists a range $h_+ < h < h_U$ where the customers inverse demand is unstable, that is, where the demand is not expected to exist at equilibrium. This region corresponds also to negative optimal prices for the monopolist: the curve $\hat{\mathcal{D}}$ is below $\mathcal{D}$. An equilibrium demand only exists beyond the point $\eta = \eta_U$, $h = h_U$, at which the monopolist's profit is zero because the optimal price vanishes ($\hat{\mathcal{D}} = \mathcal{D}$).

The critical value h_c at which the monopolist should switch from the strategy of low prices and large fractions of customers to the one with high prices and small fractions of customers

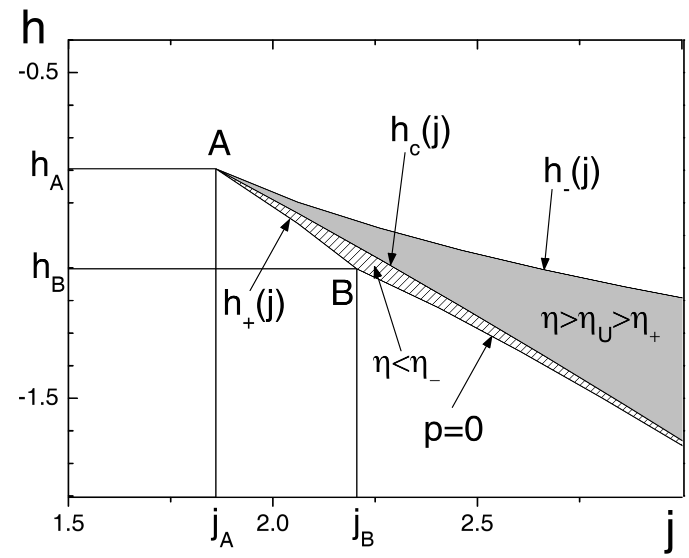

Figure 8: *Supply phase diagram in the plane* ($j = J/\sigma,\ h = H/\sigma$)*, smooth case: example of the case where the customers IWP has a logit distribution.*

when $j > j_B$ is seen to lie within the region where the customers meet the coordination problem. Since in this region the actual equilibrium of the customers system does not depend uniquely on the posted price (through the difference $\delta = h - p$), the monopolist cannot be sure to reach the optimal profit by posting the price. On figure 7 we represented the prices p_{inf} and p_{sup} (clearly, $p_{inf} > p_{sup}$) and the corresponding fractions of buyers $\eta_{inf} < \eta_{sup}$ at the value $h = h_c$, for which the corresponding profits are equal.

Figure 8 presents the phase diagram illustrated on the particular case of the logit distribution. In the shadowed region the monopolist's profit presents multiple maxima. Inside this very same parameter region, for $j > j_B$, customers meet a coordination dilemma. The critical line $h_c(j)$ where the monopolist's strategy should jump from the '+' to the '-' solution (or vice versa), is indicated. However, since for $j > j_B$ prices alone are unable to drive the customers' decision, the monopolist may not obtain the expected profit. On figure 9, the same results are presented with the alternative choice of parameters, ($\tilde{\sigma} = \sigma/J,\ \tilde{h} = H/J$) (see 2.2, (12)), with which one is looking at the model properties in term of the strength of heterogeneity (the variance of the IWP, measured in units of J).

4.3.4 Vicinity of the singular points A and B

As we have seen, the bifurcation point A plays the same role, in the monopolist's phase diagram, as the bifurcation B in the Demand phase diagram. The singular behaviour at this apex$A$ is obtained in the very same way. Developing in the vicinity of the A, at which $\widehat{\Gamma}''(\eta_A) = 0$, we obtain expressions for the boundaries of the multiple extrema region that are similar to (38), but with $\widehat{\Gamma}$ in the place of Γ, $\hat{j}$ instead of j and $\hat{\epsilon} \equiv 2\epsilon$ instead of ϵ.

It is interesting to consider more in details the vicinity of the (monopolist's) point B, where one has both $p = 0$ and marginal stability for the '+' solution. Near B, for $j > j_B$ and/or $h > h_B$, the '+' solution gives a small price value, and a value of η close to η_B. According to the expansion done for the demand, (38), we can expect a similar behaviour for the monopolist's solution: a linear increase of the price and a singular, square root, behaviour for η. Indeed at

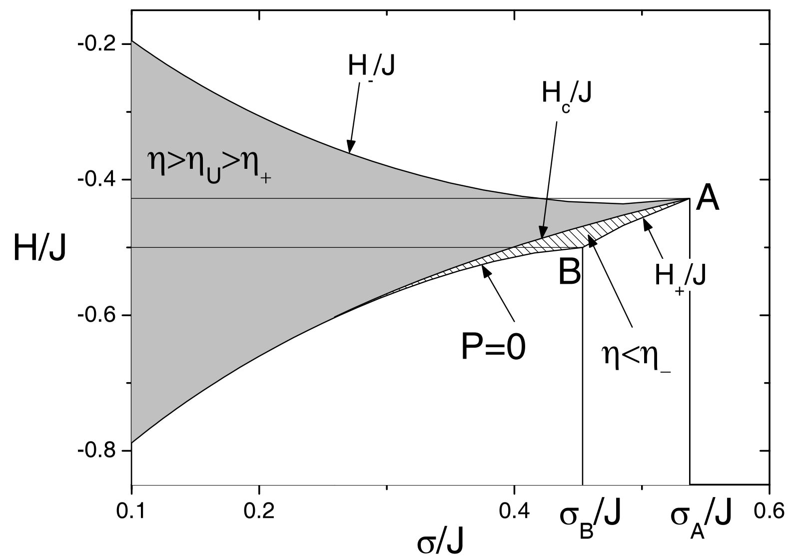

Figure 9: *Supply phase diagram, smooth case illustrated on the case of logit distribution: alternative representation, here in the plane* ($\tilde{\sigma} = \sigma/J$, $\tilde{h} = H/J$).

first non trivial order in ϵ one gets

$$\text{for } \ j = j_B,\ 0 < \epsilon \equiv h - h_B << 1 : \begin{cases} p_+ &= \epsilon, \\ \eta_+ &= \eta_B + \sqrt{\frac{2}{\eta_B\ \Gamma'''(\eta_B)}}\ \epsilon^{1/2}, \\ \Pi_+ &= \eta_B\ \epsilon + \sqrt{\frac{2}{\eta_B\ \Gamma'''(\eta_B)}}\ \epsilon^{3/2} \end{cases} \tag{77}$$

And similarly,

$$\text{for } \ h = h_B,\ 0 < \epsilon \equiv j - j_B << 1 : \begin{cases} p_+ &= \eta_B\ \epsilon \\ \eta_+ &= \eta_B + \sqrt{\frac{2}{\Gamma'''(\eta_B)}}\ \epsilon^{1/2}, \\ \Pi_+ &= \eta_B^2\ \epsilon + \eta_B\ \sqrt{\frac{2}{\Gamma'''(\eta_B)}}\ \epsilon^{3/2} \end{cases} \tag{78}$$

The '+' solution appears at B through a continuous transition for the profit Π, with a discontinuous jump for η (from 0 to η_B), and then a square-root behaviour. The latter is specific to the point B. Indeed, one can perform a similar expansion in the vicinity of the null price line ($p_+ = 0$, see equations (76) at $j > j_B, h < h_B$). Consider a point on this line with $j > j_B$. The corresponding value of η is the solution $\eta_0(j)$ of $j = j_0(\eta)$, and the value of h is $h_0(j) \equiv h_0(\eta_0(j))$. Then for $h = h_0(j) + \epsilon$, $0 < \epsilon << 1$, expansion of $p = p^s(\eta) = p^d(\eta)$ at first non trivial order in ϵ gives

$$\text{for } \ j > j_B,\ 0 < \epsilon \equiv h - h_0(j) << 1 : \begin{cases} p_+ &= \epsilon, \\ \eta_+ &= \eta_0(j) + \frac{\epsilon}{\eta_0(j)\ \Gamma''(\eta_0(j))}, \\ \Pi_+ &= \eta_0(j)\epsilon + \frac{\epsilon^2}{\eta_0(j)\ \Gamma''(\eta_0(j))}, \end{cases} \tag{79}$$

On sees on the above expansion for η how the singular behaviour at $j = j_B$ appears: as j approaches j_B^+, $\eta_0(j) \to \eta_B$, hence $\Gamma''(\eta_0(j))$ tends to zero, and thus the coefficient of ϵ in the expansion of η diverges.

4.4 Summary of the generic monopolist's model

In this section we analyzed the monopolist's profit optimization problem when there exist social influences among customers. We considered the generic case of finite variance distributions of the IWP. Depending on the strength of the social interactions the monopolist is faced to three different kinds of situations. For $j < j_A$, where the precise value of j_A, given by equation (72), depends on the details of the IWP distribution, the profit optimization gives a single price for each value of h, the average willingness of the population. If $j_A < j < j_B$, there is still a single solution for each value of h, but the monopolist's profit presents two maxima: one corresponding to few customers and a high price, the other to a large fraction of customers at a low price. Depending on the value of h and/or j, he has to select the optimal strategy. Because the customers solution in this region is a unique function of the price, the monopolist can drive the market through the posted prices, and thus earn the maximum profit. The situation is very different for $j > j_B$: since in this region the customers meet a coordination problem, the monopolist cannot drive the market though prices alone. Social interactions thus introduce uncertainty in the monopolist strategy. In particular, if $\delta_U < h < h_c$ and the customers succeed to coordinate on the Pareto dominant equilibrium (see section 3.2.4) the monopolist's solution on the low η branch (high prices and small fraction of buyers — see figure 7 —) is not realizable.

From a game-theoretic point of view, the monopolist commits himself by choosing unilaterally a price p that maximizes his profit given the aggregate result of the customer's best response to this price. Even if the players move simultaneously, this game with monopolist pre-commitment has the same structure as a two-stage game where the monopolist moves first by pre-committing the price, and the customers move second and make their choice given this price. This is a kind of hierarchical subgame-perfect equilibrium, where customers are at an N-agents Nash equilibrium at the second node, and the monopolist is in a Stackelberg position with respect to the aggregate demand at the first node. Unfortunately in the region of multiple solutions, there are two possible subgames (Nash) equilibria for the same price. Therefore, the price pre-commitment is not sufficient for selecting one of the two equilibria, each one being subgame perfect.

5 Conclusion and perspectives

The model of collective behavior considered in this paper, under the general hypothesis detailed in section 2, may be declined in both non-economic and economic contexts. In the first case, one is interested in the fraction of adopters, which in the second case corresponds to studying the demand function for an exogenous price: this is the subject of the first part of this paper, section 3. In a second part, within the market context, section 4, we discuss the profit optimization by a monopolist fixing the price.

Like in many other models in the recent literature, we consider optimizing agents making binary choices, with willingnesses to adopt that depend *additively* on an idiosyncratic part (IWP) and on the choices of other agents. The population is intrinsically heterogeneous: the IWPs are drawn from some distribution of mean H and variance σ^2. In contrast with other most studied models, here the individual willingness-to-adopt heterogeneity is frozen: it does not result from (time varying) random shocks interpreted as regression errors. In other words, each the agent's choice is deterministic with a well known (to him) IWP, and we concentrate on the aggregate behaviors. We analyze the equilibrium properties (Nash equilibria) characterized by the emergence of a collective behavior resulting from the combined effect of externalities and heterogeneity.

The analysis is done for global uniform interactions, that is, a global social influence of uniform strength, in the limit of an infinite population, through the application of the central limit theorem. Results are summarized on phase diagrams which are plots where the regions of qualitatively different collective behaviors are exhibited as a function of the model parameters. In the case of the demand, these are: δ, the average willingness to adopt (the average IWP minus the price in the economic context), and j, the social influence strength, both parameters being measured in units of the standard deviation σ of the idiosyncratic term distribution (see 2.2).

One of the main results for the demand is that, for very general IWP distributions, there is a region in the phase diagram with multiple equilibria. More precisely, if the IWP distribution is mono-modal, there are two Nash equilibria for any j larger than a distribution-dependent value j_B. For smaller values of j, the (Marshallian) demand curves are, *ceteris paribus*, downward sloping (i.e. monotonically decreasing with increasing prices). For large externality strengths ($j > j_B$), when the population average willingness to pay is small enough, the demand becomes not-monotonic (as in Becker's example [9]). This is a very general property of the model with additive externalities, and does not depend on the particular statistical distribution of the idiosyncratic preferences. We also discuss (although more briefly) the results for multimodal distributions – for which there exist several regions with multiple equilibria, with possibly more than two equlibria for some of them –, and present detailed analysis of many illustrative examples. An important contribution of this paper is to exhibit the detailed properties of the boundaries of the regions in the parameters space with multiple solutions. These are generic, in that they depend only on qualitative features of the IWP distribution.

Future work may extend the results presented in this first part of the paper in several directions. First, the individual preferences may include a stochastic (noise) term like in [35, 32, 93, 18, 13, 14, 67, 4], on top of the idiosyncratic term. Second, the present paper concentrates on the equilibrium properties – that may be considered as the "static" analysis of long term equilibria in the Marshallian tradition. Further studies should focus on the process that makes the system reach one or the other of the possible equilibria. A first study, implying revision of beliefs in an repeated choice setting, has been already done [78]: in the region with multiple equilibria, interesting complex dynamics occur with a large family of different equilibria being reached, depending on the particular learning rule used by the agents. Third, literature on marketing and studies of social psychology shows that choices very often depend on imitation effects or social influence. For example, the existence of externalities in the Communication and Information Technologies (CIT) sector is well established, and may result in a multiplicity of equilibria [75]. This may arise in other sectors also. Yet, empirical and econometric studies allowing identification of the corresponding preferences distributions and the strength of the social influence are lacking [15]. Fourth, the influence of social networks topologies deserves further attention. Ioannides [48] reported results for the 'Thurnstone model, i.e. homogeneous IWP and stochastic (logistic) choices, mainly for tree-like and one-dimensional networks with nearest-neighbor interactions. It would be interesting to explore how the phase diagrams for the model considered here - heterogeneous IWPs and deterministic choices - are affected by short range interactions. Analytical and simulation studies on the RFIM [79, 73] show that the heterogeneity introduces hysteretic effects in the dynamics, with interesting path-dependant properties (return-point memory effect [79]). A statistical method to calculate the return points exactly, starting from an arbitrary initial state, has been recently proposed [81] for the simple case of a one dimensional periodic network with nearest-neighbor interactions (called cyclic topology in [48]). The impact of such properties on both individual and collective economic behavior remains to be investigated.

The second part of the paper deals with the supply in a monopolist market. We show that the equilibrium equations accept a formalization similar to that of the demand. Here also there is a region in the corresponding supply phase diagram with two equilibria. These may exist even when customers have a single Nash equilibrium. There is a threshold in the externality strength at which the monopolist's optimal strategy switches from a high price - low fraction of buyers to one where it is better to sell to a large fraction of customers at a low price. But if the externality strength is large enough, the monopolist may not be able to make the optimal price selection because the customers' demand presents itself several equilibria. In such situation, prices are not the only driving parameter of the market. In fact, the monopolist's decision can be formally viewed as a Stackelberg game where the seller selects his optimal price knowing the best response of the customers, that are in a Nash equilibrium. Our model does not include the possibility that the seller commits himself on quantities (that is, the customers behavior depend only on their own preferences and on the other agents'behavior). It would be interesting to develop the model in this direction. Indeed, in the presence of multiple equilibria, such a public commitment might help customers to coordinate on one equilibrium, thus allowing the monopolist to determine his optimal price. In Shapiro and Varian [80] there are empirical examples in the CIT sector where the market could not develop due to the absence of commitment of at least one of the actors.

Other extensions of the results of this second part involve the study of how, with repeated choices, the long term equilibria depend on the entangled dynamics where customers and monopolist learn from each other. Finally, at least two directions deserve to be explored: the case of an oligopolistic competition and the consequences of Coase conjecture in the case of choices with externalities involving a durable good – an issue already addressed in the literature [61], but not yet in the regime where multiple equilibria exist.

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A Appendix A: Profit maximization equations

A.1 Price optimization

In order to maximize the monopolist's profit, let us define

$$\Psi(\eta, p) \equiv p^d(\eta) - p. \tag{A-1}$$

The equation $\Psi(\eta, p) = 0$ defines a curve Ψ in the plane $\{\eta, p\}$ along which π has to be maximized. Be

$$\mathbf{v}(\eta, p) \equiv (v_\eta, v_p) = (\partial\Psi/\partial p, -\partial\Psi/\partial\eta) \tag{A-2}$$

a vector tangent to the curve $\Psi = 0$ at the point (η, p).

The maximization of (46) along Ψ imposes that the directional derivative of π vanishes,

$$(\mathbf{v} \cdot \nabla)\, \pi \equiv v_\eta \frac{\partial\pi}{\partial\eta} + v_p \frac{\partial\pi}{\partial p} \; = \; 0, \tag{A-3}$$

to guarantee that the profit is an extremum. If the maximum is reached inside the support of $\Gamma(\eta)$, it must also satisfy the second order condition

$$(\mathbf{v} \cdot \nabla)\, (v_\eta \frac{\partial\pi}{\partial\eta} + v_p \frac{\partial\pi}{\partial p}) \leq 0. \tag{A-4}$$

For finite range pdfs (compact support), one has to check whether the maximum maximorum lies on one of the boundaries of the support. Remark: had we chosen to consider η as a function of p, that is $\pi(p) = p\eta^d(p)$ as done in [66], this stability condition would read

$$\frac{d^2\pi}{dp^2} \leq 0.$$

Introducing the components of $\mathbf{v}$ given by equation (A-2)

$$v_\eta \;\; = \;\; \frac{\partial\Psi}{\partial p} = -1, \tag{A-5}$$

$$v_p \;\; = \;\; -\frac{\partial\Psi}{\partial\eta} = -\,\frac{dp^d(\eta)}{d\eta}, \tag{A-6}$$

into the first order condition (A-3) gives equation (48).

Using (A-5), (A-6) and (48), the second order condition (A-4) reads

$$[-\frac{\partial}{\partial\eta} - \frac{dp^d(\eta)}{d\eta}\frac{\partial}{\partial p}]\; [-p + p^s(\eta)] \;\leq\; 0, \tag{A-7}$$

A.2 Behavior of $\widehat{\Gamma}$

We show that, under the smoothness hypothesis H2, (62) is typically true. First, for 'usual' pdfs such as the logit or the Gaussian, one can easily check that $\widehat{\Gamma}$ is indeed a monotonic, strictly increasing function of η for all $0 < \eta < 1$. Let us now consider a general pdf f under hypothesis H2. Since $\Gamma(\eta)$ is a monotonically increasing function, we have $\widehat{\Gamma}(\eta) \geq \Gamma(\eta)$ for all η. From

$$\widehat{\Gamma}'(\eta) = 2\Gamma'(\eta) + \eta\Gamma''(\eta), \tag{A-8}$$

and the fact that $\Gamma''(\eta) \geq 0$ for $\eta \geq \eta_B$ and $\Gamma''(\eta) < 0$ for $\eta < \eta_B$, we have the following inequalities:

$$\widehat{\Gamma}'(\eta) > 2\Gamma'(\eta) > 0 \quad \forall \eta > \eta_B \tag{A-9a}$$

$$\widehat{\Gamma}'(\eta) \leq 2\Gamma'(\eta) \qquad \forall \eta \leq \eta_B. \tag{A-9b}$$

Moreover,

$$\widehat{\Gamma}'(\eta_B) = 2\Gamma'(\eta_B) = 2j_B > 0 \tag{A-10}$$

From (A-9a), one has that $\widehat{\Gamma}$ is a monotonically increasing function for $\eta > \eta_B$, and by continuity of Γ', (A-10) insures that this is the case at least for η smaller than — but close enough to — η_B. However the inequality in (A-9b) does not ensure that $\widehat{\Gamma}(\eta)$ is monotonic for still smaller values of η.

Consider the behavior of $\widehat{\Gamma}'(\eta)$ for η close to 0. Clearly it depends on how fast $\Gamma(\eta)$ diverges to $-\infty$ when $\eta \to 0$, and this depends on whether the pdf $f(x)$ decays fast enough as $x \to \infty$. In the case of the logit, $\Gamma \sim \log \eta$; for the Gaussian, $\Gamma \sim -\sqrt{-2\log\eta}$; for a power law, $\Gamma \sim -\frac{1}{\eta^b}$. This suggest to consider the following smooth behavior:

$$\text{as } \eta \to 0\text{:}\ \ \Gamma(\eta) \sim -K(-\log\eta)^a \frac{1}{\eta^b}, \tag{A-11}$$

with the constant $K > 0$, $a \geq 0$ and $b \geq 0$ ($ab \neq 0$). Then, from the behavior of the derivatives of Γ at small η one gets (to leading order in η):

$$\widehat{\Gamma}'(\eta) \sim -\frac{\Gamma(\eta)}{\eta}\ [\ b(1-b) + \frac{a(1-2b)}{(-\log\eta)}\ ]. \tag{A-12}$$

Hence, for $a = 0$, that is for pdfs having a power law tail when $x \to +\infty$, if $b < 1$ (a necessary condition for the pdf to have a finite variance; see section B.2 for more details), the leading term within the square brackets in (A-12) is positive, so that, at least for $\eta \to 0$, $\widehat{\Gamma}'$ is positive. For $a > 0$, the condition is $b < 1/2$. Thus, pdfs with finite mean and variance give raise to monotonic functions $\widehat{\Gamma}(\eta)$ for small enough η. We will come back in section B.2 to the particular case of fat-tail distributions with infinite variance.

For distributions with finite variance, it remains the possibility to have a non monotonic behavior of $\widehat{\Gamma}$ in some small intermediate range, η not too small and not too close to η_B, but this would require that the pdf exhibits a sharp change of behavior — but nevertheless smooth — on a very small range of values.

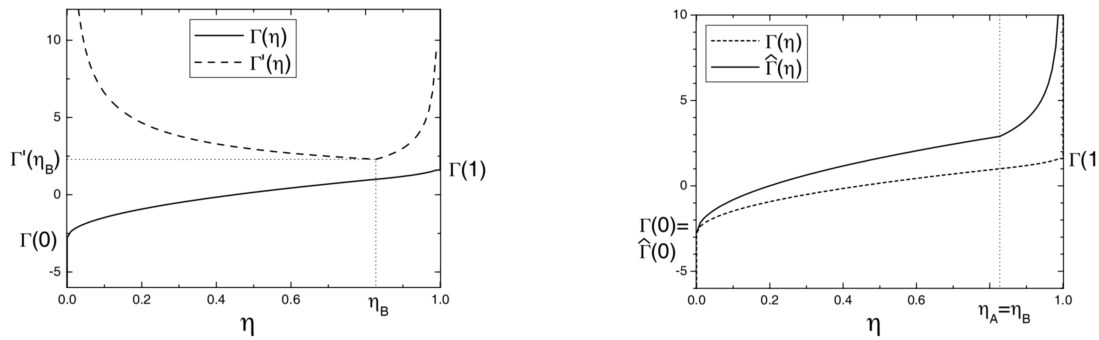


Figure 10: *Triangular pdf of unitary variance and a maximum at* $x_B = -1$. *Left:* $\Gamma(\eta)$ *and its first derivative. Right:* $\Gamma(\eta)$ *and* $\hat{\Gamma}(\eta)$ *for comparison.*

B Appendix B: Demand and supply for other distributions

In this Appendix we extend our analysis to more general distributions. We first (section B.2) relax hypothesis H5, and consider pdfs with unbounded support called *fat tail* distributions in the literature. In section B.1 we explicit the particularities introduced on the above generic results when the pdf has a bounded support. Finally, we extend our results to multimodal distributions in section B.3.

B.1 Pdfs with compact support

We consider here pdfs $f(x)$ with compact supports: $x \in [x_m, x_M]$. As before we consider the case of a pdf with a unique maximum (which may be located at one boundary). Clearly, such pdfs have finite variances. The discussion follows the same lines as that of the generic smooth distributions, except that in addition one has to pay attention to the values of Γ and its derivatives at the boundaries $\eta = 0$, $\eta = 1$. A simple uniform distribution, analyzed in [43], is a particular case where the maximum of the pdf is degenerate.

In this section, derivatives at the boundaries, like $\Gamma'(1)$ or $\Gamma'(0)$, stand for the left and the right derivative of Γ at $\eta = 1$ and $\eta = 0$, respectively. Due to the fact that the pdf strictly vanishes beyond its support, if the price is very high with respect to h there may be no buyers at all, and $\eta = 0$. On the contrary, it it is so low that the market saturates, i.e. $\eta = 1$. Consequently, there are new boundaries in the phase diagram, delimiting the regions where these solutions exist. In the next sections we first present results for the demand, and then for the supply.

B.1.1 Compact support: the demand

In the case of compact supports $[x_m, x_M]$, $\Gamma(\eta)$ increases from $\Gamma(0) = -x_M < 0$, to $\Gamma(1) = -x_m > 0$. Hence, like in the generic case of unbounded supports, Γ' reaches a minimum at η_B, and there is a critical value $j_B = 1/f_B$ beyond which multiple solutions appear. Notice that if the maximum of the pdf lies at x_m or at x_M, η_B lies at one of the boundaries of the $[0, 1]$ interval. If the pdf is symmetric — as is the case for the uniform distribution —, $\eta_B = 1/2$. The figures in this Appendix correspond to a triangular pdf with a mode at x_B inside the support. The function Γ and its first derivative is represented on figure 10.

Due to the finite range of the compact domain, there are two new particular values of δ: $\delta_0 \equiv \mathcal{D}(j; 0) = \Gamma(0) = -x_M$, independent of j, and $\delta_1(j) \equiv \mathcal{D}(j; 1) = \Gamma(1) - j = -x_m - j$. For

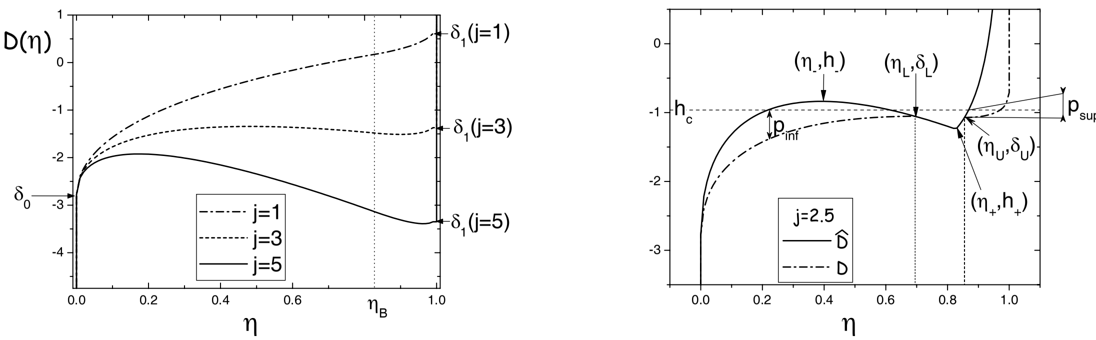


Figure 11: *Triangular pdf of unitary variance and a maximum at $x_B = -1$. Left: $\mathcal{D}(j;\eta)$ for different values of j. Right: Functions $\mathcal{D}(j;\eta)$ and $\widehat{\mathcal{D}}(j;\eta)$ showing the different boundaries of the monopolist's phase diagram parameterized by η, for a value of $j > j_B$.*

$\delta < \delta_0$, $\eta = 0$, while for $\delta > \delta_1(j)$ the market may saturate: $\eta = 1$. These extreme values of η may be reached upon finite prices only in the case of compact supports (see figure 11).

For $j < j_B$, $\mathcal{D}(j;\eta)$ is strictly increasing on $]0,1[$, and is also invertible: for any δ in $]-x_M, -x_m - j[$, equation (20) has a unique solution $\eta^d(\delta)$.

One can easily check that $j < j_B$ implies $j \le x_M - x_m$, so that $-x_M < -x_m - j$: $\mathcal{D}(j;0) < \mathcal{D}(j;1)$. In the particular case where f is the uniform distribution, one has precisely $j_B = x_M - x_m$. For $\delta < \delta_0$, $\eta = 0$, bounding the region where the market exists. For $\delta > \delta_1(j)$ the market saturates. In between, the fraction of buyers/adopters is a monotonic function of δ.

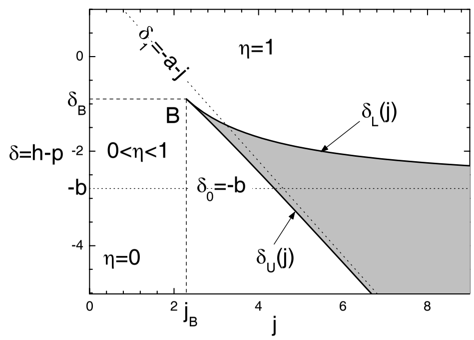
For $j > j_B$ one has two stable solutions whenever $\delta_U(j) \le \delta \le \delta_L(j)$. Due to the existence of the extreme solutions $\eta = 0$ and $\eta = 1$, the analysis is more cumbersome than for infinite supports. If the maximum of the pdf lies inside the support, the solutions $\eta_L(j)$ and $\eta_U(j)$ of equation (29) lie in $]0,1[$ and $\delta_U(j)$ and $\delta_L(j)$ both satisfy $\mathcal{D}' = 0$. On increasing δ from $-\infty$, there is no demand until $\delta = \min\{\delta_0, \delta_U(j)\}$. If $\delta_0 < \delta_U(j)$, when δ increases beyond δ_0 the demand becomes finite and remains unique provided that $\delta_0 < \delta < \delta_U(j)$. For $\delta > \delta_U(j)$ we enter the region of multiple solutions. On the other hand, if $\delta_0 > \delta_U(j)$, the system steps directly from the no-demand solution into a region where a finite demand equilibrium coexists with the no-demand one. In both cases, the coexistence region exists for $\delta_U(j) \le \delta \le \delta_L(j)$. Notice that the high η solution may correspond to either a fraction of buyers strictly smaller than 1 (if $\delta_L(j) < \delta_1$) or to saturation (if $\delta_L(j) > \delta_1$). The values of j at which the extreme solutions appear satisfy $\mathcal{D}(j_0;\eta_U) = \delta_0$ and $\mathcal{D}(j_1;\eta_L) = \delta_1(j)$ respectively.

If the pdf has a maximum at one of the boundaries of its support, either η_U or η_L coincide with one of the extreme solutions.

Summarizing, the customers phase diagram for pdfs with compact supports have two supplementary lines with respect to that with unbounded supports. They indicate the boundary of the *viability region*: no market exists below this line, and the *saturation* boundary, above which all the customers are buyers. Figure 12 presents an example corresponding to a triangular pdf of unitary variance, with a maximum at $x_B = -1$.

B.1.2 Compact support: the supply

Consider now the profit optimization. The function $\hat{\Gamma} = \Gamma + \eta\Gamma'$ increases from $\hat{\Gamma}(0) = \Gamma(0) = -x_M$ to $\hat{\Gamma}(1) = \Gamma(1) + \Gamma'(1) \ge -x_m$. The inflexion point η_A of $\hat{\Gamma}$, that determines the value of $j_A = \hat{\Gamma}'(\eta_A)/2$, may lie inside the support or at one of its boundaries. Besides this, which has to be handled correctly in each case, it is straightforward to establish the phase diagram of the monopolist.



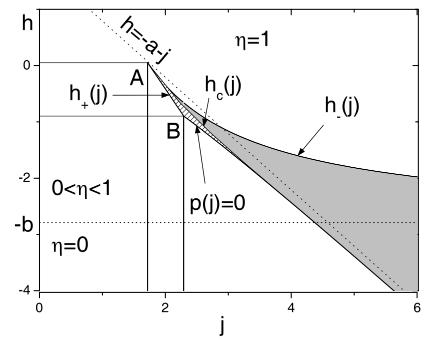


Figure 12: *Triangular pdf of unitary variance and a maximum at $x_B = -1$. Left: Customers phase diagram. Right: Monopolist's phase diagram. h_c is the line where the monopolist's strategy has to change between a low to a high price*

In particular, for $h < h_0(j) \equiv \Gamma(0) = -x_M$ there is no market whatever the price. Notice that a similar case was already met with the fat tails distribution: if $h < 1$ for $j < j_A = 1$ and $h < h_+(j)$ for $j > j_A = 1$, there are no buyers at any price. The viability limit is represented on the phase diagram by the line $h_0(j)$, which is independent of j. Only if $h > h_0$, equation (57) has non-vanishing solutions for η. The saturation region, on the other hand, exists for $h > h_1(j) \equiv \Gamma(1) + \Gamma'(1) - 2j = -x_m + \Gamma'(1) - 2j$. In the region $h_0 < h < h_1$, the fraction of buyers satisfies $0 < \eta < 1$. Since $\Gamma'(1) = 1/f(-x_m)$ we have: $h_1(j) = \Gamma(1) + 1/f(x_m) - 2j$.

For $j < j_A$, the optimal price is uniquely determined by the value of h, and the monopolist has a single strategy: the optimal price is $p^*(j,h) = \eta^*[\Gamma'(\eta*) - j]$, with $\eta^*(j,h)$ solution of (57).

For $j > j_A$ there are in principle two strategies, like in the case of unbounded supports, although here it may arise that one of them does not exist. Due to the viability limit, for $j > j_B$ the solution with largest fraction of buyers is bounded by the zero-price line, that satisfies $h = \delta$.

Figure 12 presents the monopolist's phase diagram for a triangular pdf of unitary variance and mode at x_B.

B.2 Pdfs with fat tails

Fat-tail distributions are characterized by the fact that the pdf f has such a slow decrease at large values of x that $a = 0$ and $b \geq 1$ in equation (A-11).

A particular example of a fat tail distribution is a pdf with a power law decrease, which for large x behaves like:

$$f(x) \sim x^{-(1+\mu)} \tag{B-1}$$

with $\mu \geq 0$. Then, for small η, $\Gamma \sim -\frac{1}{\eta^b}$ with $b = 1/\mu$, so that $b \geq 1$ means $\mu \leq 1$. For $\mu < 1$, not only the variance but also the mean value of the random variable x is infinite.

For fat-tails distributions one has to look at finite size effects: it is no more possible to take directly the large N limit and make use of the central limit theorem: quantities like $(1/N)\sum_i G(x_i)$ for any function G will be dominated by rare events, that is by the largest values encountered in the population of (large but finite) size N. There is, however, no difficulty in doing this analysis: the results are obtained by doing *as if* the pdf had a finite support, the upper bound x_M being given as an increasing function of N (for an introduction to statistics with fat tails, see e.g. [16]).

Since we consider compact supports in the next section, we concentrate here on the marginal case $\mu = b = 1$, which can be analyzed as a limiting case of distributions with infinite support.

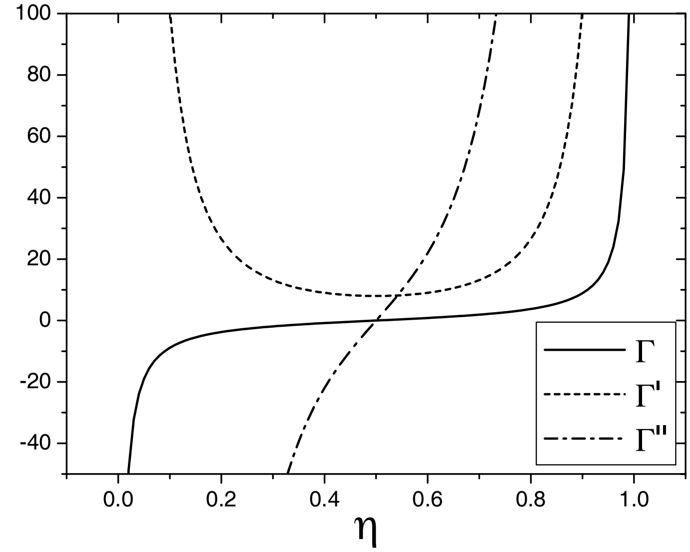


Figure 13: Γ *function and derivatives corresponding to the function (B-2).*

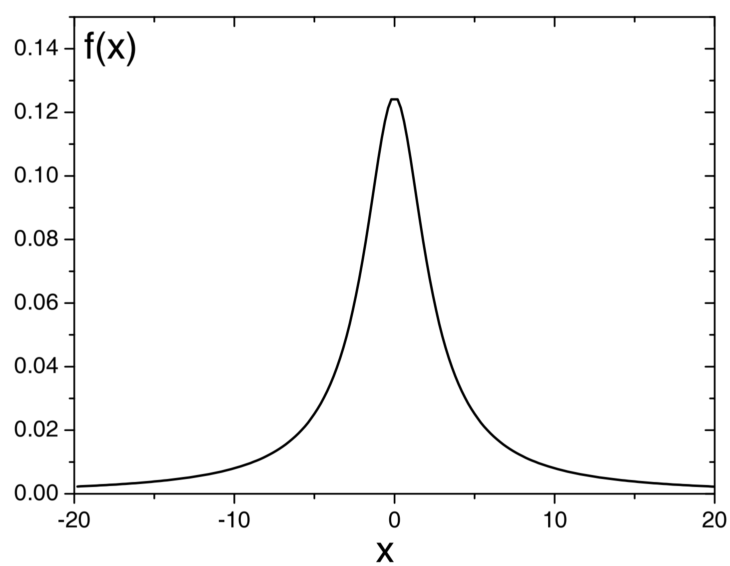


Figure 14: *Pdf corresponding to equation (B-4).*

For $\mu = 1$, $f(x)$ does not have a finite variance. Then, the value of σ that defines the normalized variables (10) may be any (finite) measure of the width of the distribution, as for example, the value of x at which $f(x)$ is equal to half its maximum. Let us discuss this marginal case on a simple example (see figure 13:

$$\Gamma \equiv -\frac{1}{\eta} + \frac{1}{1-\eta}, \tag{B-2}$$

corresponding to the cumulative function:

$$F(z) = \frac{1}{2} - \frac{1}{z} + \mathrm{sgn}(z)\sqrt{\frac{1}{z^2} + \frac{1}{4}}. \tag{B-3}$$

The corresponding pdf is,

$$f(x) = \frac{1}{x^2}[1 - \frac{2}{\sqrt{4+x^2}}], \tag{B-4}$$

as represented on figure 14. Since this is a symmetric distribution, $\eta_B = 1/2$, and one finds (see equations (27), (28) and (34)) that the point B in the customers phase diagram is $j_B = \Gamma'(\eta_B) = 8$, $\delta_B = -4$. Notice that, like for any monomodal distribution (satisfying thus hypothesis H1), $\Gamma(\eta)$ is convex for $\eta > \eta_B$ and concave for $\eta < \eta_B$, with as before η_B being the inflexion point. The supply function has thus the generic behavior described in section 3 even for fat-tail distributions. Thus, we concentrate on the monopolist's profit optimization.

Since from (B-2) and (58) we have

$$\widehat{\Gamma}(\eta) = (1-\eta)^{-2} \tag{B-5}$$

it easy to find the supply bifurcation point A beyond which there are multiple relative optima. It corresponds to the minimum of $\widehat{\Gamma}'(\eta) = 2(1-\eta)^{-3}$. Thus, for this fat-tails distribution, $\eta_A = 0$ and

$$A \equiv \{j_A = 1, h_A = 1\} \tag{B-6}$$

Notice that this is not a relative but an *absolute* minimum, so that the considerations following equation (66) do not apply, and one has to consider explicitly $\widehat{\mathcal{D}}(\eta) = \widehat{\Gamma}(\eta) - 2j\eta$. Figure 15 presents $\mathcal{D}(\eta)$ and $\widehat{\mathcal{D}}(\eta)$ for $j = 11 > j_B$.

Clearly, for $j < j_A \equiv 1$, the (absolute) minimum of $\widehat{\mathcal{D}}$ is $\eta = 0$, with $\widehat{\mathcal{D}}(0) = 1$. Thus, for $j < j_A$ the optimal solution is a single valued function of h and j. Whatever the value of $j < j_A$, finite fractions of buyers — and hence finite profits — can only be expected if $h \geq 1$; otherwise $\eta = 0$. In the vicinity of $\eta = 0$, to lowest order in $h - h_+$ we have:

$$\text{for } \; j < j_A, \; h << 1 : \begin{cases} \eta & = \frac{h-1}{2(1-j)}, \\ p & = 2\frac{1-j}{h-1} = \frac{1}{\eta}, \\ \Pi & = 1 + \frac{(h-1)^2}{4(1-j)} = 1 + (1-j)\eta^2. \end{cases} \tag{B-7}$$

Thus, when $h \to 1^+$, $\eta \to 0^+$, $p \to \infty$, but $\Pi \to 1$. To understand this result one has to bear in mind that the present analysis is done in the limit of a large number of customers ($N \to \infty$). In fact, for finite populations, the smallest non-vanishing value of η is $\eta = 1/N$, which gives $p = N$: for large finite populations, for $h = 1$, there is an optimum obtained by selling a single unit of the good to a single agent (hence $\eta = 1/N$), the one with the largest willingness to pay, at a price of the order of N (hence Π of order 1). At $h \to 1^+$, for increasing values of j (but $j < j_A$) the optimum moves monotonically to lower prices and larger fractions of buyers, while the profit increases.

For $j > j_A$, the minimum of $\widehat{\mathcal{D}}(\eta)$ lies at $\eta_+ = 1 - 1/j^{1/3} > 0$. Thus, a solution with a finite fraction of buyers exists whenever $h > h_+(j) = 3j^{1/3} - 2j$, and this is the only possible

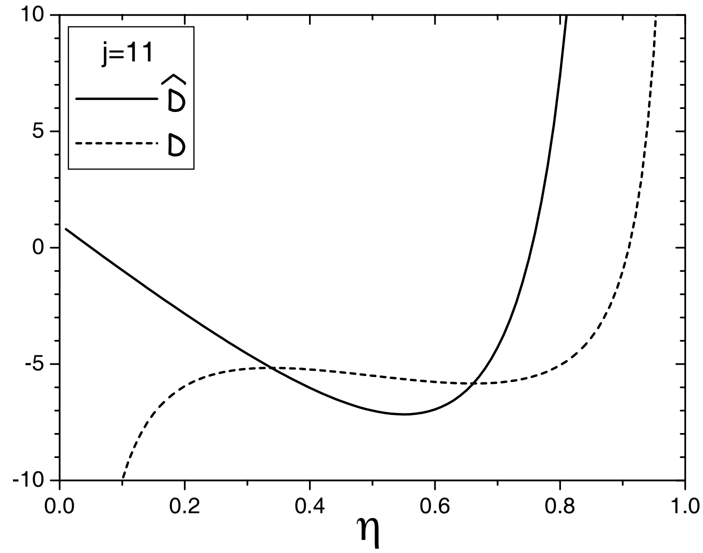


Figure 15: $\mathcal{D}(j;\eta)$ *and* $\widehat{\mathcal{D}}(j;\eta)$ *for* $j > j_B = 11$, *corresponding to the fat-tail distribution of equation (B-4).*

solution with finite η since for $0 \leq \eta < \eta_+$ the slope of $\widehat{\mathcal{D}}(\eta)$ is negative (the maximum condition $\widehat{\mathcal{D}}'(\eta) \geq 0$ is not satisfied). Thus, when $j > j_A$ the monopolist has only one optimum: to sell the good to a large fraction of buyers. For each value of $j > j_A$, the profit is a monotonic increasing function of h. As in the generic case, when $j > j_B$ there is a line of null price bounding the region of economically viable solutions.

To summarize: the point A lies on the line $h = 1$. For any given $j < j_A = 1$, upon increasing h there is a continuous transition from a solution $\eta = 0$ to one with $\eta > 0$, at which $\Pi = 1$. When $j > j_A$, there is a first order (discontinuous) transition from $\eta_- = 0$ to the solution with $\eta_+ > 0$, with $\Pi(\eta_+) \geq 1$. Thus, in contrast with the case analyzed in section 4.2, as soon as the solution with finite η exists, the monopolist's optimal strategy is to sell the good: there is no region with multiple extrema. The resulting phase diagram is shown on figure (16).

The optimal prices and the corresponding profits as a function of h are represented on figure 17 for two different values of j. For $j = 11$ the optimal price vanishes at $h = -7.1617$ (not shown on the figure).

B.3 Aggregate demand for multimodal pdfs

B.3.1 Smooth pdfs: generic properties

In this section we consider the behavior of the application $\delta \to \eta^d(\delta)$ in the case of a smooth multimodal pdf with support on $]-\infty, +\infty[$. The discussion section 3.2, based on convexity arguments, can be extended to describe the phase diagram for the aggregate demand in the multimodal case.

The minimal hypotheses we consider are the following.

- HA0. The pdf $f(x)$ is continuous with a finite number $K \geq 2$ of x-values, $-\infty < x_B^K < x_B^{K-1} < ... < x_B^1 < \infty$, for which f has a (possibly local) bounded maximum,

$$f(x_B^k) < \infty \quad k = 1, ..., K. \tag{B-8}$$

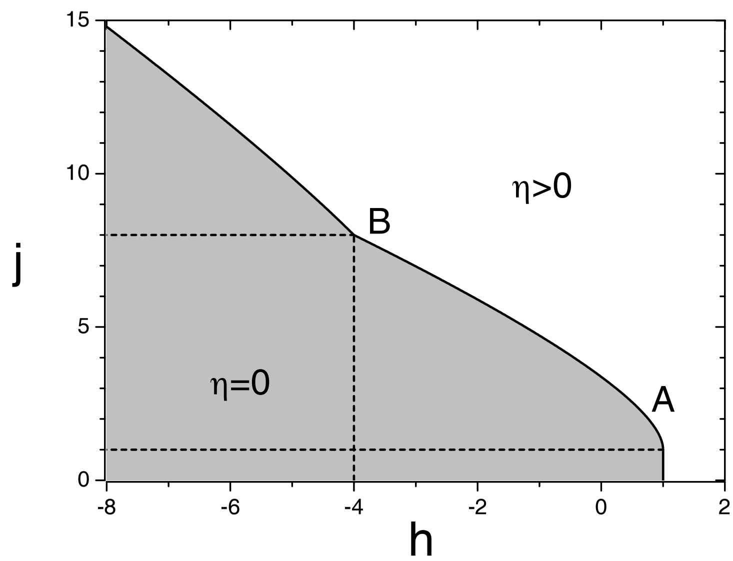


Figure 16: *Supply phase diagram for a pdf with an infinite variance ($b = 1$).*

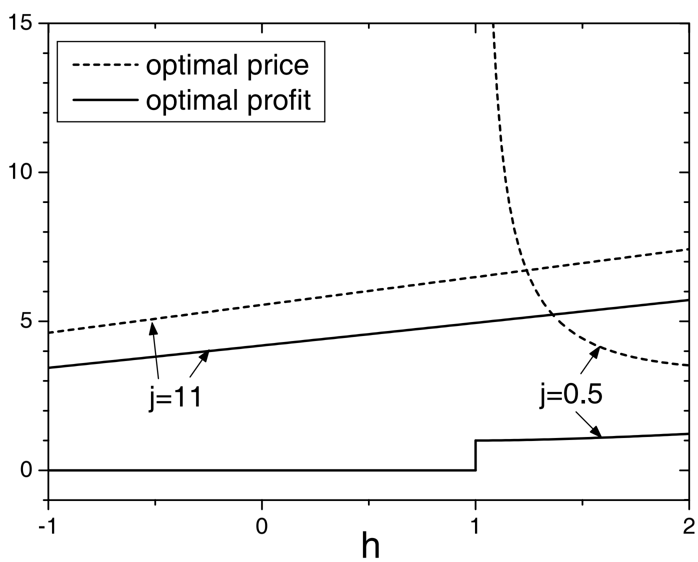


Figure 17: *Optimal prices and corresponding profits as a function of h for different values of j, for a pdf with an infinite variance ($b = 1$).*

For simplicity we assume also that the pdf is not constant on an any interval of finite length. Actually, the discussion can be easily extended to less regular pdfs (in particular piecewise continuous pdfs), and pdfs constant on some intervals, but to keep the discussion shorter will leave that to the interested reader (in the case of a monomodal pdf, see the discussion on compact supports, and for the bimodal case see also below, section B.3.2, the singular case of a distribution composed of two Diracs).

- HA1. When considering smoother functions, we will assume f to be twice continuously differentiable, so that in particular it has a zero derivative at every maximum and every minimum.

Let us denote by x_C^k the location of the minimum between x_B^{k+1} and x_B^k. We assume $f > 0$ everywhere on its support except possibly at some minima, and $f(x)$ goes to zero as $x \to \pm\infty$. By convention we set $x_C^0 = +\infty$ and $x_C^K = -\infty$ (and we may write $f(x_C^0) = f(x_C^K) = 0$).

In the monomodal case, we have seen that the critical value of j for the appearance of several solutions is $j_B = 1/f(x_B)$. Here we will see that the relevant critical values are

$$j_B^k \equiv \frac{1}{f(x_B^k)} \qquad k = 1, ..., K \tag{B-9}$$

and also

$$j_C^k \equiv \frac{1}{f(x_C^k)} \qquad k = 1, ..., K-1 \tag{B-10}$$

(and it will be convenient to define as well $j_C^0 = j_C^K = \infty$).

Consider now the inverse demand at a given value of j. The ensemble of equilibria $\delta^d(\eta)$ for $\eta \in [0, 1]$ is the subset of the ensemble of solutions of (20) for which δ increases (p decreases) as η increases. As for the monomodal case we study the function of η defined by (20) for any given j, $\delta(\eta) = \mathcal{D}(j; \eta)$. By continuity of the function Γ, $\delta(\eta)$ is a continuous function of $\eta \in [0, 1]$. As $\eta \to 0$, $\delta \to -\infty$, and as $\eta \to 1$, $\delta \to +\infty$. Increasing η from 0, $\delta(\eta)$ increases. Similarly, decreasing η from $\eta = 1$, $\delta(\eta)$ decreases. Since $f(x)$ is continuous, $\Gamma(\eta)$ is continuously differentiable, with $\Gamma'(\eta) \equiv d\Gamma(\eta)/d\eta = 1/f(x)$ at $x = -\Gamma(\eta)$. Hence $\Gamma'(\eta)$ has (local) minima at values of η given by

$$\Gamma'(\eta_B^k) = \frac{1}{f(x_B^k)} \qquad k = 1, ..., K \tag{B-11}$$

and (local) maxima at values of η given by

$$\Gamma'(\eta_C^k) = \frac{1}{f(x_C^k)} \qquad k = 1, ..., K-1 \tag{B-12}$$

For a smooth enough pdf, the η_B^k and η_C^ks are inflexion points for Γ. Note that for any $k = 1, ..., K$, $\eta_C^{k-1} < \eta_B^k < \eta_C^k$.

The most important remark is that Γ is strictly concave on every interval $]\eta_C^{k-1},\ \eta_B^k[,\ k = 1, ..., K$, and strictly convex on every interval $]\eta_B^k,\ \eta_C^k[,\ k = 1, ..., K$. Then as η varies on $[\eta_C^{k-1},\ \eta_B^k]$, the function $\mathcal{D}(j; \eta) = \Gamma(\eta) - j\eta$ has, at some value $\eta_L^k(j)$, a maximum $\delta_L^k(j)$ which is by definition the Legendre transform of Γ restricted to $[\eta_C^{k-1},\ \eta_B^k]$. Similarly, on $[\eta_B^k,\ \eta_C^k]$, $\mathcal{D}(j; \eta)$ has, at some value $\eta_U^k(j)$, a minimum $\delta_U^k(j)$, the Legendre transform of Γ restricted to $[\eta_B^k,\ \eta_C^k]$.

Depending on the value of j compared to the values j_B^k, j_C^k, these min and max may be reached either at a boundary of an interval, or in the interior. More precisely:

$$j < j_B^k, \quad \eta_L^k = \eta_U^k = \eta_B^k \tag{B-13}$$

$$j_B^k < j < j_C^k, \quad \eta_B^k < \eta_U^k < \eta_C^k \tag{B-14}$$

$$j_B^k < j < j_C^{k-1}, \quad \eta_C^{k-1} < \eta_L^k < \eta_B^k \tag{B-15}$$

$$j_C^k < j, \quad \eta_L^{k+1} = \eta_U^k = \eta_C^k \tag{B-16}$$

(and η_U^k increases from η_B^k to η_C^k as j increases from j_B^k to j_C^k, whereas η_L^k decreases from η_B^k to η_C^{k-1} as j increases from j_B^k to j_C^{k-1}). In the case of a continuously differentiable pdf, every Legendre transform $\eta_\Lambda^k(j),\ \Lambda = L, U$ satisfies the marginal stability equation,

$$\frac{\partial \mathcal{D}(j;\eta)}{\partial \eta}|_{\eta=\eta_\Lambda^k(j)} = 0. \tag{B-17}$$

One should note that $\eta_{U,L}^k$ and $\delta_{U,L}^k$ depend on j (and on the function $\Gamma(.)$), but not on h or p.

Now for $j < j_B \equiv \min_k j_B^k$, every min and max are reached at the corresponding value η_B^k: this means that there is no intermediate regime with a decreasing behavior of $\delta(\eta)$ as η increases, hence $\delta^d(\eta) = \mathcal{D}(j;\eta)$, uniquely defined, is a continuously increasing function of $\eta \in [0,1]$. For $j > j_B$, there is at least one k where the maximum $\delta_L^k(j)$ is reached for $\eta = \eta_L^k(j) < \eta_B^k$, and the minimum $\delta_U^k(j)$ is reached for $\eta = \eta_U^k(j) > \eta_B^k$, so that there is at least one finite interval of η on which the function $\mathcal{D}(j;\eta)$ decreases with η, and thus does not correspond to an economic equilibrium. Hence the demand $\eta^d(\delta)$ has at least two branches.

In the plane (j,δ), the boundaries of the multiple solutions regions are thus given by the fonctions $\delta_\Lambda^k(j) = \mathcal{D}(j;\eta_\Lambda^k(j)),\quad \Lambda = L, U$, which are the graphs of all the branches of the Legendre transform of Γ. By construction of the Legendre transform, every branch $\delta = \delta_U^k(j)$ is a concave curve, and every branch $\delta = \delta_L^k(j)$ is a convex curve, and, under the smoothness hypothesis HA1, along each branch $\Lambda = L, U$,

$$\frac{d\delta_\Lambda^k(j)}{dj} = \frac{d\mathcal{D}(j;\eta_\Lambda^k(j))}{dj} = -\eta_\Lambda^k(j). \tag{B-18}$$

Recall that η_Λ^k is the value of η for the solution which is marginally stable on this boundary.

These boundaries can be easily drawn for any distribution making use of a parameterization by s (or equivalently $x \equiv -s$): from the basic equations $\eta = 1 - F(-s)$ where F is the cumulative of the pdf f, $s = \Gamma(\eta)$, and $\Gamma'(\eta) = 1/f(-s)$; with the marginal stability condition (B-17) which gives $j = \Gamma'(\eta)$, the locus of marginal stability is then given in the plane (j,δ) by the parameterized curve

$$\text{for } x \in support(f),$$

$$j = 1/f(x) \tag{B-19}$$

$$\delta = -x - \frac{1-F(x)}{f(x)} \tag{B-20}$$

This is this representation that we have used to draw the phase diagram, figure 19, for the particular example of the bimodal distribution shown on figure 18.

The domain of multiple solutions can then be described as follows. The phase diagram is a kind of superposition of diagrams associated to mono-modal phase diagrams, every maximum (every 'bump' in the pdf) k being responsible of the appearance of a domain of multistability: when j becomes larger than j_B^k, a continuous solution split into two solutions, with a lower solution $\eta^d(j,\delta) \le \eta_L^k(j) < \eta_B^k$ and $\delta \le \delta_L^k$, and an upper one with $\eta^d(j,\delta) \ge \eta_U^k(j) > \eta_B^k$ and $\delta \ge \delta_U^k$ (see figure 19). When j becomes larger than j_C^k, this bump is no more 'seen'. Since a minimum of the pdf, if not as a boundary, is in between two maxima, such an intermediate solution may exist either because of one bump or the other - or both.

The branch $\delta = \delta_U^k(j)$ has thus as left end point, $B^k \equiv (j_B^k, \delta_B^k = \mathcal{D}(j_B^k;\eta_B^k))$, and as right endpoint (if j_C^k is finite), $C^k \equiv (j_C^k, \delta_C^k = \mathcal{D}(j_C^k;\eta_C^k))$. B^k is the merging point of δ_U^k and δ_L^k, and C_k the merging point of δ_U^k and δ_L^{k+1}. Since δ_L^k and δ_L^{k+1} must be both above δ_U^k, these two branches must intersect one another for some value of $j = j_{BC}^k$ between j_B^k and j_C^k: there is thus coexistence of three solutions in the triangular-like domain bounded below by δ_U^k (or

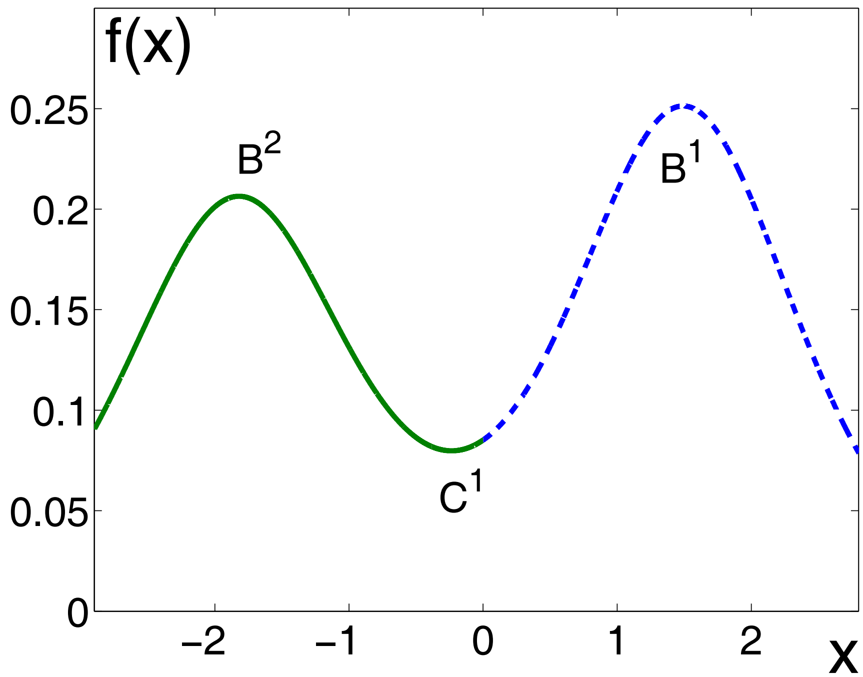


Figure 18: *An example of bimodal pdf.*

$\max(\delta_U^k,\ \delta_U^{k+1})$ if B^k is below the branch δ_U^{k+1}), and above by δ_L^k for $j \leq j_{BC}^k$, and by δ_L^{k+1} for $j \geq_{BC}^k$.

In the smooth case (HA1), at every bifurcation point B^k, resp. C^k where two boundaries merge, according to B-18 there is a common slope $-\eta_B^k$, resp. $-\eta_C^k$.

One may say that the pdf is probed at different scales for different values of j. Consider the graph $y = f(x)$. Every maximum below the line $y = 1/j$ is not seen (it does not change the structure of the solution), whereas a set of maxima higher than $1/j$, but joined by minima where f is still higher than $1/j$, is seen as a single global bump. This gives in particular that for $j > j_B$, the number of solutions is equal to one plus the number of times the line $y = 1/j$ cut the graph $y = f(x)$ at points where f is increasing. Note that this does not give the number of solutions for a given value of δ. On figure 20, two pdfs are shown; the intersection of the graph $y = f(x)$ with the line $y = 1/j$ gives the structure of the demand at this particular value of j (in the case illustrated on the figure, the demand has 3 solutions for the two pdfs).

B.3.2 A degenerate case: 2 Dirac

Let us consider the particular case of an IWP distribution given by two Delta pics: $x_i = \pm x_0$ with equal probability ($x_0 = 1/\sqrt{2}$ since the variance of f is normalized to 1). For $j = 0$, one has clearly $\eta = 0, 1/2$ or 1 depending on $\delta < -x0$, $-x_0 < \delta < +x_0$ or $\delta > x_0$. For $j > 0$, obviously η can still take only these three values. One gets easily the domain of existence and stability of these solutions, $\eta = 0, 1/2, 1$, by direct inspection of the equation (13). The resulting phase diagram is shown on figure 21.

This phase diagram for a singular distribution can also be understood by comparison with the predicted phase diagram for a continuous distribution. In the present case, the two maxima have equal height, $+\infty$, which gives $j_B^1 = j_B^2 = 0$, in agreement with the fact that boundary

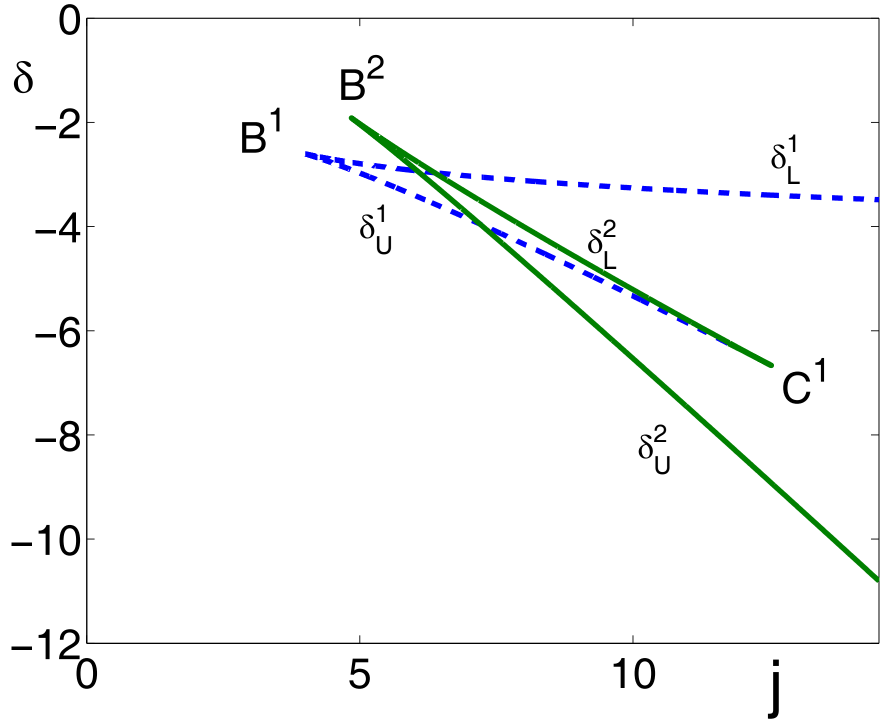


Figure 19: *Phase diagram (aggregate demand) for the case of the smooth bimodal pdf shown on figure 18.*

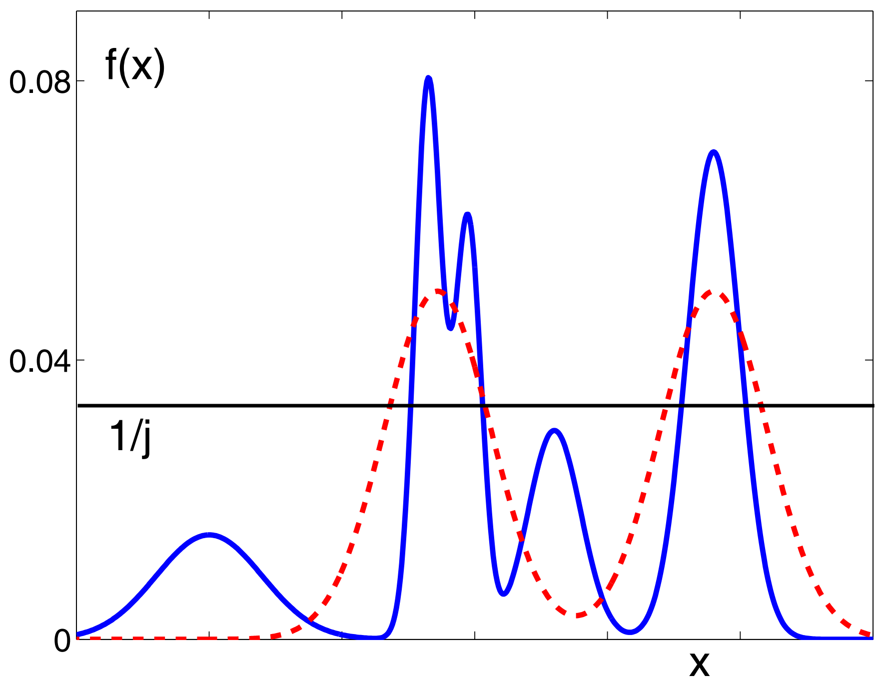


Figure 20: *Examples of multimodal pdfs. At a given value of $j = J/\sigma$, the qualitative properties are obtained by looking at the intersection of the horizontal line $y = 1/j$ with the graph of the pdf, $y = f(x)$: for the particular value of j corresponding to the horizontal line on this figure, the two pdfs lead to the same qualitative properties of the Demand.*

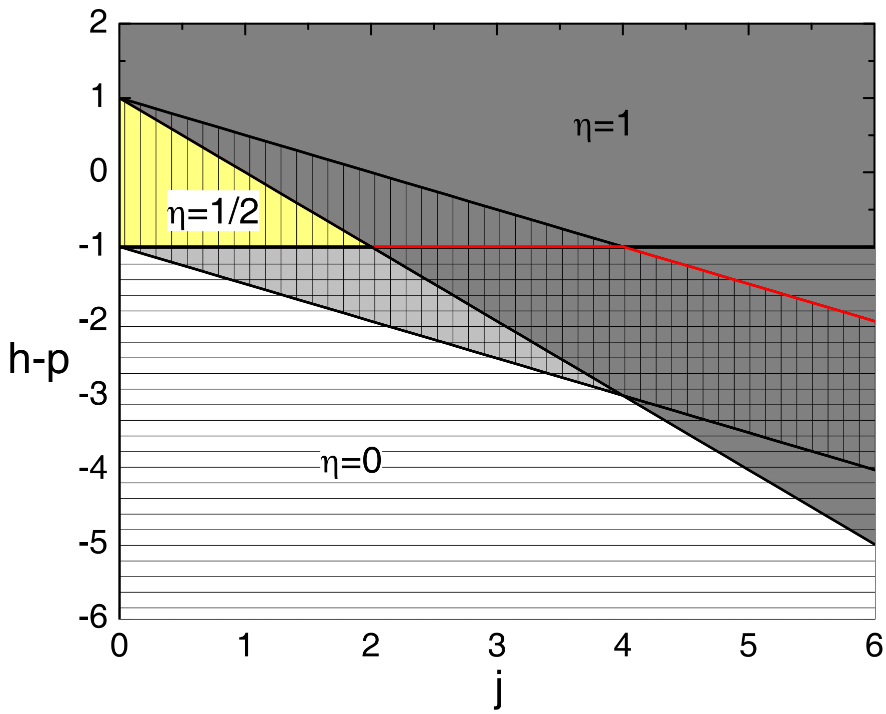


Figure 21: *Phase diagram (aggregate demand) for the case of a bimodal pdf composed of two Dirac peaks.*

lines meet at $j = 0$. The minimum between the two maxima is at $f = 0$, hence $j_C = \infty$: the domain of stability of the intermediate solution $\eta = 1/2$ extends to infinity, as it is the case whenever a minimum is at $f(x_C^1) = 0$. The marginal stability lines are straight lines - hence, marginally concave and convex curves -, with slopes $0, 1/2$ and 1 corresponding to the values of the solution marginally stable on the boundary, in agreement with (B-18). Since here there is no continuity in the demand at the singular points $B^1 = (0, -1), B^2 = (0, 1)$, two branches do not merge with a common slope: besides the fact that the demand can take only three values, this is the only place where the non smoothness of the pdf gives a feature of the phase diagram qualitatively different from what is obtained for a smooth pdf.